ANGLIA RUSKIN UNIVERSITY
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PURE SCIENCE AND LOGICAL TALES: WHEN LOGIC SEEMS STRANGER THAN FICTION. A FRAMEWORK FOR AN ILLUSTRATIVE INTERPRETATION OF LEWIS CARROLL'S LOGIC.

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A thesis in partial fulfilment of the requirements of Anglia Ruskin University for the degree of PhD.

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Acknowledgements

I would particularly like to thank my tutor Martin Salisbury who, after almost 5 years of supervising this PhD, has enabled me to make this project a reality and bring it to completion.

My interest in drawing began in the studio of the painter Jean Edelmann (1916-2008), for whom I have a special thought today.

It was then with the illustrators Rocco (cartoonist for LE MONDE newspaper) and Janet Woolley that I discovered illustration.

Rocco turned my attention to storytelling, engraving and poster art by supervising my illustration study of « 40 Tales of the World » .

Subsequently, Janet Woolley encouraged me to develop new areas of research in illustration and to explore various drawing techniques.

Finally, my thanks go to Dr. Peter de Meza for his careful reading of the thesis and to my family, and in particular to my sister Aude Sainte Cluque who gave me the idea of creating pop-ups, an idea without which all pop-up games would not have seen the light of day in this form.

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Abstract

This research is motivated by the desire to establish a bridge between logical tales and pure science, sciences and visual arts, and to determine a link between rationality and fantasy, two *a priori* antagonistic universes.

In the context of children's illustration, it has two aims. The first is to illustrate what seems to be non-illustratable, such as abstract concepts and complex reasoning. The second objective is to investigate whether visual arts can be elevated to the status of metalanguage that can help illustrate scientific languages and participate in discoveries in this field.

Pedagogically, the difficult issue is not to employ artistic language to teach children to read, write and count but rather to think, question and reason.

My "practice-based research" method is centred on the image/text ratio in educational books and games. By adding reasoning to this ratio, I propose a methodology for using creative artwork to express scientific concepts. The comprehension of the text is essential and is combined with eight criteria, including aesthetics and ethics. For this purpose, I use puzzles, counters, cards and instruction manuals.

The study begins with Lewis Carroll's two works, the *Game of Logic* and *Symbolic Logic*. As a storyteller, logician and mathematician, he created a universe of discourse to teach children the rules of argumentation both amusingly and entertainingly. Then, I focus my attention on ancient and modern logic which proceeds from Aristotle and the Stoics to computational thinking.

As a result, I have created several pop-up games that show how abstract concepts can be used into practice.

Other researchers might well be able to apply this method to other reasoning models, for example, the inductive and analogical models of experimental sciences.

Introduction

Strange as the conjunction may seem, the purpose of this research is to establish a connection between pure science and logical tales and, in particular, to conjure up literary writers of nonsense such as Lewis Carroll or Ionesco to highlight the link between nonsense and logic. This leads curiously but necessarily to a major issue: How can abstract concepts and complex reasoning be illustrated through visual arts?

Lewis Carroll's motivation for writing *Alice's Adventures in Wonderland* (1865) and *The Game of Logic* (1886) was not solely to entertain children with wordplay, nonsense and fantasy but also to facilitate them in comprehending some of the fundamental concepts of the 'Universe of discourse'. These concepts pose significant challenges to any illustrator who wants to do more than embellishing a text, especially if one wishes to use images or drawings to help the understanding of the strange reasoning that lies behind the apparent nonsense of Carrollian prose. The very nature of these concepts means that the traditional image/text ratio cannot satisfactorily fulfil the role. Under the name of Lewis Carroll, Charles L. Dodgson is dealing with at least three different elements, the word, the image and, through storytelling and tales, reasoning in mathematics and logic.

Many of the concepts and logical reasoning outlined in this thesis will be very familiar to those educated in these disciplines. However it will seem much less obvious to others. Thus, the first aim of this research is to make them more easily accessible through creative visual arts, games and illustrated instruction manuals. This question is pedagogically important. It consists of establishing a link between art and science by the use of creative visual arts, as did Leonardo da Vinci with his diagrams, drawings and prototypes. This requires overcoming preconceived ideas, including those in the field of illustration. It has often been said that 'We think of book illustrations as pictures which elucidate and decorate a page of printed text', nevertheless illustration has had to evolve. It is now omnipresent and implemented in many literary, educational works, in books and magazines, advertisements and posters. When it comes to making abstract concepts understandable through visual arts, a question arises. How to illustrate what seems to be non-illustratable? Pure sciences and in particular logic are very abstract. If the problem of illustrating complex concepts in pure science is solved, the method can probably be used in less abstract areas. To show that the solution is not inaccessible, I can give an example of an abstract concept, 'the principle of contradiction'.

A contradiction between words, sentences, ideas, arguments and discourses is a complex notion. It is used in quotidian life and in many other fields. This abstract concept did not prevent Dickens and Doré from illustrating it in their own way. Charles Dickens, in his writing and Gustave Doré in his drawings, were able to depict contradictions and paradoxes which existed in Victorian time. Dickens, a tireless advocate of children's rights and education for all, highlighted the contradiction between, reality and the fantasy that prevailed in his time. In *Hard Times: For These Times* (1854), he showed the paradoxes of the Industrial Scientific Revolution that Gustave Doré was able to show through his images in a series of 180 engravings ('London: A Pilgrimage', 1872).

Dickens described the technical progress of James Watt's steam engine used in textiles and transport, the first underground railway in the world, the garden parties, the horse auctions. At the same time, he highlights the homeless sleeping beneath the bridges with above them a galloping industrialisation that accentuated social divisions. In essence, progress was something that one could rejoice in or deplore. This concept of contradiction highlighted here has several possible interpretations and therefore, possible illustrations. In his dialectic, Hegel considers that this principle is 'the motor of History', whilst Aristotle makes it a tool to judge the consistency and validity of reasoning. This research focuses on this second aspect of knowledge.

The pedagogical role of illustration has evolved because of the significant place that children have acquired in society. This reflects the changing attitudes and laws towards children. Schooling became compulsory. The aim was to offer them a proper education through reading, pictures and games. The second half of the 19th century is broadly considered to be the golden age of illustration in both Europe and the United States. The fables, legends and fairy tales destined for children became illustrated more often. In Britain, famous illustrators such as Caldecott, Crane, Greenaway, Lear and *The Book of Nonsense*, Rackham and the tales of the Brothers Grimm, John Tenniel and Lewis Carroll's *Alice Adventures in Wonderland* significantly influenced the collective imagination, especially that of children. In France, with artists like Daumier, Grandville, Doré, Riou, de Montaut, Barbant, Benett, Manet and Degas, illustration is elevated to the rank of art. Jules Verne's illustrated novels excitingly combine science and fiction.

As a result, the concept of illustration itself evolved. During this period of effervescence in technology, scientific education was to be found in a wonderland where instruction and amusement were the keys to evolution. At this time, the boundaries between the occult and official sciences were not yet clearly drawn, nor were the boundaries between competent scientists, amateurs and charlatans. Therefore, some primary issues come to mind. Can we trust tale illustrators to explain scientific theory? Is it not paradoxical to want to associate two antagonistic universes such as rationality and fantasy? Fairy tales remain a fable, a playful fiction, a subjective false truth; a vast lie, some would say. Science – in the search for truth and proof – is considered objective and timeless. Yet, is the frontier between fairy tales and science completely impermeable? What is the impact of science on fairy tales, and vice versa? The idea of linking art and science is not new. It has often been acknowledged that Leonardo da Vinci's (1452–1519) flying machine sketches were the precursor of the helicopter. Since Einstein's theory of relativity, the postulate of wormholes has become a reality. With 'Quantum teleportation', *The Time Machine: An Invention* (Wells, H. G., 1895) becomes a subject of interest again. Therefore, where is the border between fiction and reality, tales and science? Another question is to investigate whether the "creative visual arts" can contribute not only to illustrate universal scientific languages but also participate in discoveries in this field. It is not just an issue of popularising science, but of giving access to its way of thinking and reasoning.

To establish visual conjunction between pure science and fairy tales, paradoxically between logic and nonsense and more generally between science and art, the method focuses here on the image/text ratio and the understanding of complex and abstract ideas. The theoretical and practical way in which the method is conceived could *a fortiori* extend to other less abstract and formal fields of knowledge.

The result of this initially academic research reported in this thesis is the creation of nine prototypes including seven pop-up games with instruction manuals, hereinafter titled "Booklets". The illustrated Booklets allow children to engage rapidly with the games (by reading the "quick start" section), and for older children, to go further, in learning the first fundamental principles of logic. This research consists of three parts divided into a total of nine chapters detailed in the table of contents.

Part I. Objectives and methods

Chapter I

Academic context

The first chapter answers three questions. Why choose logical tales and science, and more precisely the relationship between science and art, as a thesis subject? Why Lewis Carroll? Why logic? It specifies the objectives to achieve. This approach mainly uses the "Art-Based Research Practice" method for illustration and the deductive method of pure sciences for logic (distinct from empirical sciences).

1.1 Research objectives and expected outcome

The thesis aims to make abstract concepts in the field of pure sciences accessible to as many people as possible through drawing. The expected result is the illustration of various key concepts, especially in the discipline of logic. The research does not focus on the psychology of tales or their literary approach, but on the logic of tales. This logic can be seen through the work of Lewis Carroll in *The Game of Logic* (1886) and *Symbolic Logic* (1896). These works are generally not illustrated. However, this is not a thesis on Lewis Carroll, his life and work, as exists elsewhere. Lewis Carrol's work is seen in this research as a means of giving evidence of abstract logical concepts through drawing, keeping in mind that the principles of logic – 'the Art of Thinking' – are according to Aristotle at the foundation of science.

1.2 Bridging the art-science divide

To make children think, question and reason through drawings and tales, requires bringing together arts and sciences. It involves designing and experimenting with a set of methodological tools used both in illustration and in other disciplines. The methodological framework, detailed below, is essentially that of 'Practice-Based Research' in visual Art (Sullivan, 2004), but also that used by researchers in other disciplines such as theatre, cinema, music or cartoons (Leavy, 2015). It is in this context that I intend to develop my drawing technique to enable the illustration of abstract concepts. One of the particularities of illustration is its language. It is used to illustrate other languages: poetry, tales, fables, novels, science fiction and everyday language. Educational books can be confronted with abstract and symbolic languages such as those of pure sciences. Hence the problem: how can abstract languages be illustrated using the language of illustration? The challenge consists of developing drawing techniques that can teach children concepts which are not easy to understand and are not very entertaining for them. It even becomes a necessity to use illustration when we are talking to a young public who cannot yet read. Abstract concepts are notions, ideas, principles that are not directly perceivable by the senses. They are only intelligible through reason and thinking.

It is the case for metaphysical concepts that are beyond the physical. They often seem stranger than fiction. The principle of contradiction remains a good example of this. To contradict oneself in a discourse or argument is generally regarded as evidence of inconsistency. Nonetheless, how can we illustrate the idea of not contradicting ourselves? For conducting this type of academic research, the university framework is certainly the most suitable. It is in line with my future objectives and interests in illustration, research and teaching.

1.3 Personal context

1.3.1 Motivation

The question that naturally arose was whether my education would enable me to solve the problems I raised regarding the illustration of abstract concepts in the pure sciences. Several elements led me to believe that it was not impossible to meet this challenge.

Firstly, my educational background is one based on a science/mathematics Baccalaureat although that is not especially relevant here.

The second most important motivation was my meeting with Jean Edelmann who was both a painter and a mathematician. When studying, I learned many of my courses whilst drawing with Edelmann (1916–2008), former student of the École Polytechnique of Paris. He explained the mathematical lessons by replacing numbers and symbols with forms and colours that I had to identify. Initially, this was intended to be entertaining. However, soon we realised it was an effective way of attracting attention in complex and abstract concepts. This certainly enabled me to take an interest in Oliver Byrne's work (1847) for my subject. Byrne succeeded in translating Euclid's geometry and equations into symbols and colours in the manner of the painter Robert Delaunay or Mondrian. It gave me the desire to link science and art.

A third motivation is the interest in illustration and teaching. Not knowing what I was going to study after my degree, Edelmann's thinking left its mark on me. 'If drawing is what you desire to do every day and if you can do it for hours and hours without noticing it, then draw.' It is what led me to take Fine Arts and anatomical drawing courses at the Beaux-Arts de Paris, before continuing my education in illustration in Paris, London and now Cambridge.

A fourth element led me to become interested in the work of Lewis Carroll. From childhood, tales and myths captivated me, and they still do. It was inevitable for me to be interested in Lewis Carroll's stories and to want to illustrate them, as several illustrators have done before (ch. 2.6 below). However, I always had the feeling there was something else to discover to the text of this mathematician and logician; which made me hesitant to illustrate any of his work without a better understanding.

A fifth and final element is the following. Having obtained a science/mathematics Baccalaureat and illustrated children's stories, it became possible to combine mathematics with painting. After all, is that not what Leonardo da Vinci did by applying his mathematical knowledge of the golden ratio for perspective drawing? This requires going beyond illustration and taking an interest in the visual arts as a whole.

My work with Edelman has served me well here. He painted on different materials e.g. glass, cardboard, large wall paintings, subway tickets. It encouraged me to draw on and with various mediums. Hence the idea of using puzzles, playing cards and pop-ups in the thesis. However, this requires mastering different techniques.

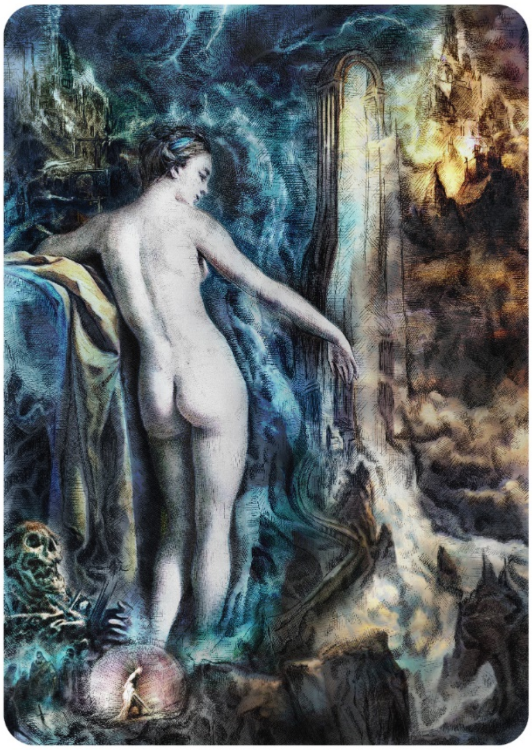
1.3.2 Drawing: From ballpoint pen to computer

During my training, I developed several drawing techniques that were useful here. Having started drawing by hand, my project in Cambridge was to employ the drawing techniques that the computer allows, with all the possibilities, advantages and disadvantages that comes with it. Below represents the evolution of my drawing techniques, which is summarised under four headings. These were then used to design and illustrate the puzzles, pop-ups and to build game aprons that form a major part of my thesis.

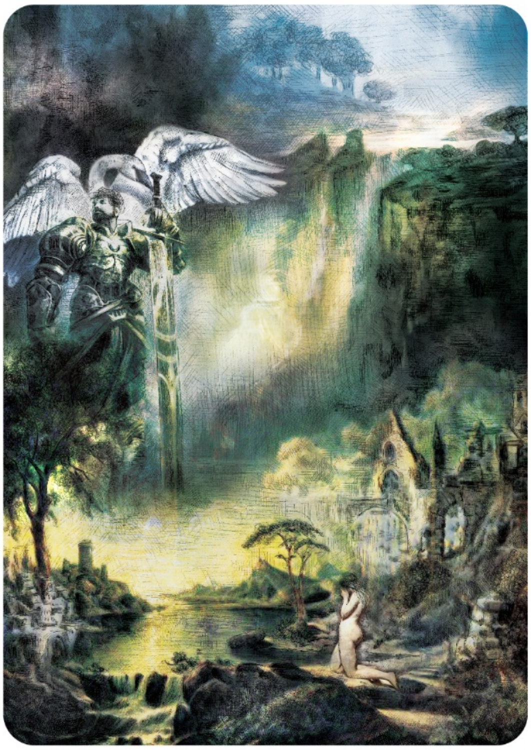
Firstly, the classical drawing's technics and the ballpoint pen

I first learned what is called classical drawing techniques with its four canonical painting modes from the Renaissance and respectively named: *Cangiante, Chiaroscuro, Sfumato* and *Unione*. From reproduction of masterpieces, sculptures, life drawings, portraits and with an emphasis on geometry such as linear perspective, I was taught the transformation and replacement of colours with analogous ones (*Cangiante*), contrast and volume (*Chiaroscuro*), blurry and smooth drawing transitions (*Stumato*), and clear and vivid drawing transitions (*Unione*). In 2011 in Shanghai, I undertook a postgraduate course in drawing and perfected my reproduction techniques. Whilst there, I was offered the opportunity to teach drawing to children which has helped me to understand how children learn and think. This has been especially important in helping design these games and pop-ups which after all are aimed at children as well as adolescents.

PERSONAL ILLUSTRATIONS



PERSIFAL –
OPERA BY RICHARD WAGNER



LOHENGRIN –
OPERA BY RICHARD WAGNER

Being less attracted to different painting techniques, I did not use colours in my drawing for some time, preferring the use of pen and ink and pencil. In 2008, I started drawing with a black ballpoint pen. The result is reminiscent of engraving but is obtained with less complexity (and lower cost). Designed to be practical and inexpensive, the ballpoint pen continues the tradition of wood engraving by Thomas Bewick (1753–1828) or steel engraving by William Blake (1757–1827) but in a far more economical fashion.

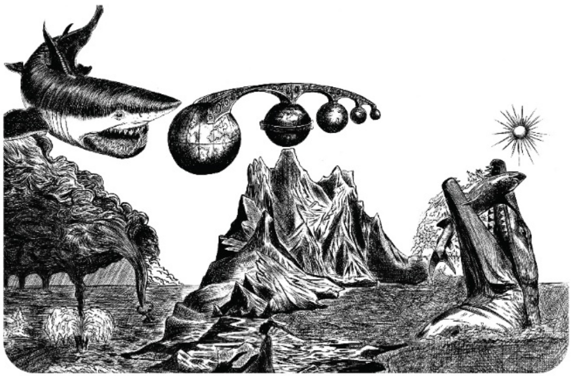
PERSONAL ILLUSTRATIONS

A TABLE –
FANZINE

PETER AND THE WOLF –
SYMPHONIC FAIRY TALE
BY SERGUEÏ PROKOFIEV

If the origin of illustration is found in ancient Egypt, engraving and mass-printed illustration have enabled its wide diffusion, with artists such as Daumier, Grandville, Doré, Caldecott, Greenaway, Crane and, closer to home, Ardizzone (1900–1979) with his crosshatching method, or Sendak (1928–2012) and his masterpiece *Where the Wild Things Are*. This line hatching technique has several constraints. It requires concentration and does not allow for the error. Any major modification or new experimentation requires the entire drawing be repeated. It is through practice rather than theory that I learnt to create a volume from hatching, to create a light source, a particular texture, to represent time and space.

PERSONAL ILLUSTRATIONS



CROSS WORLDS –
FANZINE

SINBAD THE SAILOR -
FROM THE NOVEL
BY ALEXANDRA DUMAS

Some authors, such as Matt Rota (2015), see in this technique a brain activator and gives several examples of artists who use the ballpoint pen technique[1].

Secondly, from ballpoint pen to computers

For my master's degree in Illustration, I was introduced to certain software and I tried to develop my ballpoint technique by associating the computer with it to introduce colour into my drawings. Instead of using coloured ballpoint pens, I started by scanning my black ballpoint drawings to add colour on the computer. This allowed me the possibility to experiment with colours without the necessity of composing volumes and shades that were already present in the scan of the black and white drawing. Initially, I tried creating my colours with the software, and I soon realised, it would be more intuitive for me to obtain my colours using a brush and watercolours on a separate sheet of paper. I then used the computer to merge my ballpoint pen drawings with colour samples I had created separately.

PERSONAL ILLUSTRATIONS

THE DIVINE COMEDY – DANTE ALIGHIERI

INFIERNO PURGATORIO PARADISIO

Thirdly, from sketches to vectors

Those initial illustrations, shown below and drawn by hand, could not be selected because they involved a long drawing process that was not adapted for this project that has required the creation of around 300 illustrated playing cards.

[1] Such as Jonathan Bréchignac (France), Dina Brodsky (New York, US), Joo Chung (New York, US), Dominique Dawn Clement (New York, US), Chamo San (Spain), Vangilbergen (Germany), cited by Matt Rota (2015).

PERSONAL ILLUSTRATIONS

STUDY FOR THE SQUARE OF OPPOSITION (OR LOGICAL SQUARE) – PLAYING CARDS

Becoming confident with the computer, I started creating my drawings directly on the computer. A significant advantage of the vector drawing is that printing can be brought to the desired format by geometric transformations and can be duplicated in less than a second. Besides, the computer allows experimenting, without the need to recommence the illustration. To design everything required to produce a game (Cards, tokens, illustrated dice), it appeared to be the appropriate technique for me. However, the more I exploited the advantages offered by the software, the less aesthetic my drawings were, the composition became flat, and the characters lacked expression.

PERSONAL ILLUSTRATIONS

STUDY FOR THE SQUARE OF OPPOSITION (OR LOGICAL SQUARE) – PLAYING CARDS

PERSONAL ILLUSTRATIONS

STUDY FOR THE SQUARE OF OPPOSITION (OR LOGICAL SQUARE) – BOARD GAME

For someone who was taught Renaissance values, I had to go back to what I was most comfortable with; drawing. My hand-drawn sketches were then scanned on the computer solely for the purpose of vectorization. Some colours were adopted directly in the software, but most of them, especially the complex colour gradations used for board games, were created again in watercolour on a piece of paper, then scanned and finally vectorised.

PERSONAL ILLUSTRATIONS

The problem that arises in the thesis is that it is a question of illustrating reasoning and not simply the arguments (premises) that compose it. To give an overview of the problem on which the thesis is based, it can be expressed through an example: How to illustrate the following story of Lewis Carroll's Crocodilus which is a dilemma that goes back at least to the Stoic logic of ancient Greece?

TABLE 1

THE TRAGIC STORY OF CROCODILUS

A Crocodile had stolen a Baby off the banks of the Nile. The mother implored him to restore her darling. "Well," said the Crocodile, "if you say truly what I shall do, I will restore it: if not, I will devour it".

"You will devour it!" cried the distracted Mother.

"Now," said the wily Crocodile, "I cannot restore your Baby: for if I do, I shall make you speak falsely: and I warned you that, if you spoke falsely, I would devour it."

"On the contrary", said the yet wilier Mother, "you cannot devour my Baby: for if you do, you will make me speak truly, and you promised me that, if I spoke truly, you would restore it!" (We assume, of course, that he was a Crocodile of his word; and that his sense of honour outweighed his love of Babies.)

To illustrate the problem, Lewis Carroll uses the tree method:
Either a or a'
if a then c'
if a' then c

a: the Mother speak truly
a': the Mother speak falsely
b: the Crocodile keeps his word
c: the Crocodile devours the Baby
c': the Crocodile restores the Baby (he cannot devour the Baby)
c and c' together is absurd.

Hence the following chart:

b

if a' | if a

then c | then c'

Problem: What conclusion can be drawn if the mother says: "You will devour the Baby?"

Lewis Carroll's conclusion is: "Whatever the Crocodile does, he breaks his word." Since the Crocodile cannot get out of this dilemma, which is similar to the famous Liar's dilemma of the Greek philosopher Eubulides of Miletus (5th century BC.), the Crocodile only has the possibility to follow his instinct. He will eat the child, because it is in its nature. This is what Lewis Carroll concludes: "His sense of honour being thus hopeless of satisfaction, we cannot doubt that he would act in accordance with his second ruling passion, his love of Babies!"

From: Lewis Carroll, 1896, *Symbolic Logic. Part II, Advanced.* Reprint by William Warren Bartley, 1977, Book XIV, chap. II, Classical Puzzles, the dilemma: p. 425 and Lewis Carroll's conclusion: pp. 436–437, and illustration of a Crocodile from Lewis Carroll's *Sylvie and Bruno* (1889, chap. 16, A Changed Crocodile, illustrated by Harry Furniss).

The difficulty here can be summarised as follows: It is not simply a question of drawing a crocodile, as Furniss illustrates so well in *Sylvie and Bruno* (1889), but of illustrating the dilemma itself, which is a metalanguage, that is to say a language that speaks about itself. Before deciding what to illustrate, I began by illustrating the premises and the conclusion (the language) of a syllogism, as in the following example of a Lewis Carroll sorite (i.e. a sequence of several syllogisms whose conclusion must be found).

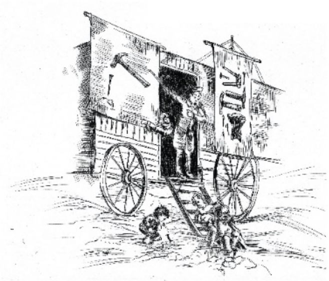
TABLE 2
PERSONAL ILLUSTRATIONS

A CAROLLIAN STORY OF BABIES AND ILLOGICAL CROCODILES

Lewis Carroll sorite:

1. Babies are illogical;
2. Nobody is despised who can manage a crocodile;
3. Illogical persons are despised.
   Conclusion: Babies cannot manage crocodiles.

From: Lewis Carroll, 1896. *Symbolic Logic.* Reprint 2015, New York and Berkeley Enterprises: Dover Publications, Inc., text p. 112 and solution pp. 157, n° 1 and 132; and respectively, Hermann, ed. 1992, text p.176 and solution p.188, n° 1.
Note. Lewis Carroll uses two methods to solve this problem: the diagrams method (Dover 2015, solution p.132) and the *Method of Underscoring* (Dover, p. 157).



1. Babies are illogical

2. Nobody is despised who can manage a crocodile

3. Illogical persons are despised

Conclusion: Babies cannot manage crocodiles

However, these drawings were not enough to illustrate the reasoning (the metalanguage) that allows to move from the premises to a valid conclusion. This led me to approach the problem of illustration in a different way which is set out in the thesis.

Fourthly, from paper to the computer to paper

Often, in illustration, when the drawing is completed, it does not necessarily mean the whole creative process is over. Printing remains an essential part of this process, and if I am well aware of this, I would never have thought at the beginning of this research that I would end up collaborating with printers. Foremost, I had to prepare my work for printing (Colours, size, resolution, formats, cut marks, bleed, cut outline…) and also decide on the choice of papers. These steps may seem obvious to most illustrators but become less when the objective is to create very large pop-up board games at an affordable price, with cards, tokens, dice, figurines, that are water-resistant, tear-proof and safe for children. The printing process used in this research has taken up a significant part of the project. Close collaboration with the printer[2] was essential, and many tests were carried out over two years. While most of the initial ideas were realised, many others had to be rethought and adapted so that they could be printed, with the printing machines available to us. Therefore, all creations are unique and would probably not be reprinted and built in the same way if I had to remake them.

PERSONAL ILLUSTRATIONS

[2] Florian Delavignes, 2C Print – Cluses. France.

1.3.3 Geometric drawing practice to understand abstract concepts

Given the broad range of possibilities offered by the computer, a question emerges. If one can draw geometric figures with a computer, should one completely ignore geometry and its formulae such as Pythagoras' theorem, the golden ratio, or the determination of the centre of gravity of a triangle? Some architects claim they can construct an ideal pyramid by eye alone (such as the glass pyramid of the Louvre in Paris) without knowing the golden ratio, the Pi number and the geometry. Others claim the opposite. The debate as to whether the Egyptian architects who built the Great Pyramid of Cheops in Giza knew the golden ratio and the number Pi is far from over. My experience in building the Square of Opposition (Game 4) and the Venn diagram (Game 5) is that it was easier to use Pythagoras's theorem and the formulae of geometry than hoping for the software to build by chance the geometric shapes I wanted. When I made playing cards and the game board to visually represent the mechanics of the Venn diagrams, I realised how useful it can be to know the theorems and formulae of Euclidean geometry before using a computer program. It avoids a lengthy trial and error when building complex geometric shapes.

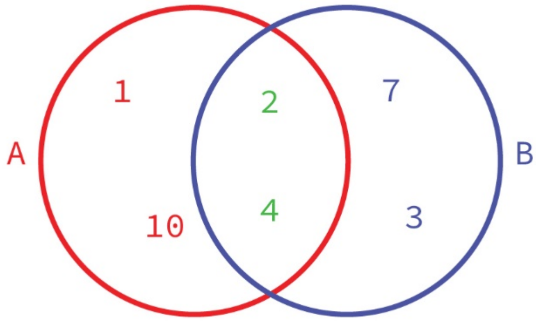
In order to realise the difficulties of design, it is necessary to recall the problem. The objective of Game 5, Venn's diagrams, was to build a game board allowing visually to find the conclusion of a syllogism (two arguments or premises and a conclusion)[3] by using the simple displacement of counters on a game board. Venn's diagrams are often used either to classify objects or to calculate probabilities with numbers. For example, in study guides for 11 to 14-year-olds (CGP, 2014, pp. 94–95), they are operated to classify numbers or objects, but not primarily to solve syllogisms.

[3] A syllogism is 'an argument that has exactly two premises and one conclusion' (Lee, 2017, p. 314). If the definition of Aristotle, the father and inventor of the theory of the syllogism is taken as precise: (*Organon, Prior Analytics*, Book I, 20): 'A syllogism is a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.' Translated by McKeon, 2001, p. 66, and translated into French by Tricot (2001, pp. 4–5) from Latin (Ciceron and Quintilien): '*syllogismus est oratio in qua consensis quibusdam et concessis aliud quid quam concessa sunt, per ea, quæ concessa sunt, necessario conficitur*'.

TABLE 3

VENN DIAGRAM (NUMBERS)

E = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {1, 2, 4, 10} and B = {2, 3, 4, 7}



Venn diagram used to classify numbers:
among the set E of numbers from 1 to 10,
the two sets A and B have the numbers 2 and 4 in common.

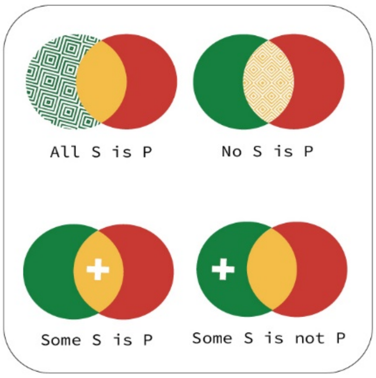
However, in Game 5, it is not a question of numbers placed in Venn circles, but of words (or groups of words) that grammatically occupy the position of a subject and predicate in a sentence. The sentence that unites the subject (noted S) and the predicate (P) through the verb (copula) is called a 'proposition', which means that it can be true or false. Depending on the two criteria retained, quality and quantity, a proposition can be affirmative or negative, universal or particular. What in a shortened language can be written: All S is P, No S is P, Some S is P, Some S is not P. This was summarised in the Middle Ages in the Square of Opposition in four letters A, E, I, O which can be represented by colours, circles and counters. In game 4, in the form of a battle game with playing cards, children will have the opportunity to learn to distinguish these four forms of propositions coded in the Middle Ages with the letters A, E, I, O, from Latin: A and I as in *affirmo,* E and O as in *nego*. Then they will be able to distinguish between opposite propositions (A and E, I and O) and contradictory propositions (the diagonals A and I, E of the Square of Opposition).

TABLE 4

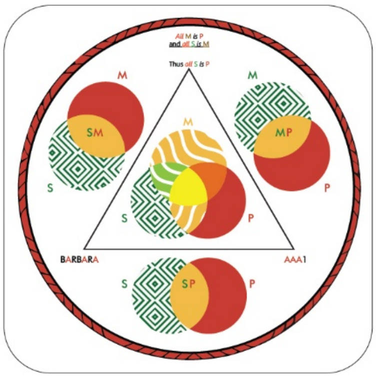
THE SQUARE OF OPPOSITIONS USING VENN DIAGRAMS,
WOODEN AND COLOURED PUZZLE PIECES

A. Universal Affirmative: All porcupines are talkative.
E. Universal Negative: No porcupines are talkative;
i.e. porcupines are not talkative.
I. Particular Affirmative: Some porcupines are talkative.
O. Particular Negative: Some porcupines are not talkative.

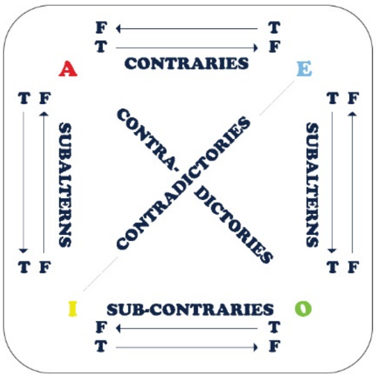
(From examples by Lewis Carroll,1896. *Symbolic logic*, ed. 2015, p. 33.)



Venn diagram method



Venn diagrams game



Square of opposition

The pedagogical problem here is to enable children to use the Venn Digrams in an entertaining and useful way to test the validity of reasoning through the visual arts. From two arguments, called premises, what valid conclusion can be drawn? With this issue in mind, here is the aim of the game. Consider two players. The first player draws a card (Darii) on which the answer is written.

He asks the second player: What can be deduced from the following two arguments (premises)[4] ?

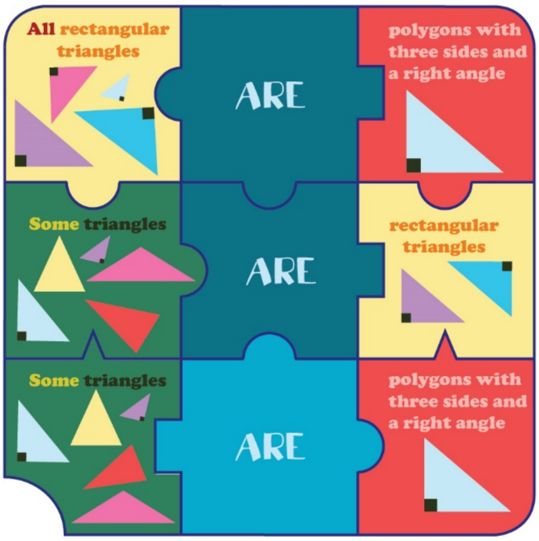
---

[4] In Game 1, the second player will find the answer by reconstructing the Darii puzzle:
'Some triangles are polygons with three sides and a right angle.' While learning concepts and definitions: polygons, triangles, sides, rectangles, children will also learn to differentiate between universal propositions (All) and particular cases (some). Not all polygons are triangles. The quadrilateral is a polygon with 4 sides, the pentagon is a polygon with 5 sides, etc. Not all triangles are rectangles, some are isosceles (2 equal sides), others are equilateral (3 equal sides), etc.

PERSONAL ILLUSTRATIONS

Starting from:
1. All rectangular triangles are polygons with three sides and a right angle,
2. and some triangles are rectangular triangles.
3. What valid conclusion can be drawn?

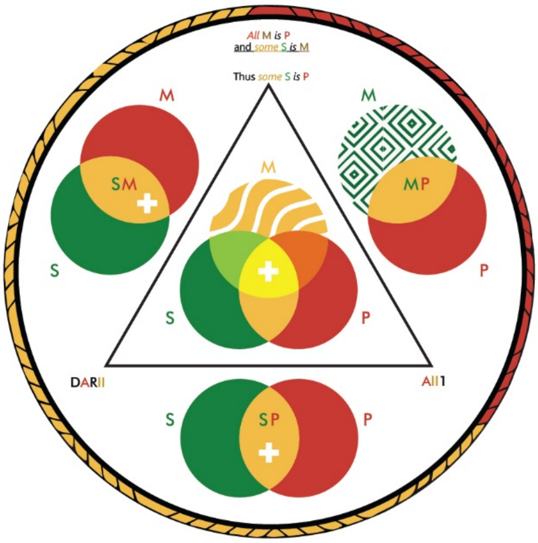
CARD FROM GAME 1



PUZZLE ASSOCIATED WITH THE CARD

Now it is a question of finding the answer using Venn diagrams, which is a visual means of arriving at a valid conclusion. The second player will find the answer by correctly placing and moving counters on a game board, as is done in chess, for example. To check the correct answer, the placement and movement of the counters are reproduced on a playing card held by the first player. This playing card must also be constructed in such a way that it visually gives the answer to the problem posed.

PERSONAL ILLUSTRATIONS



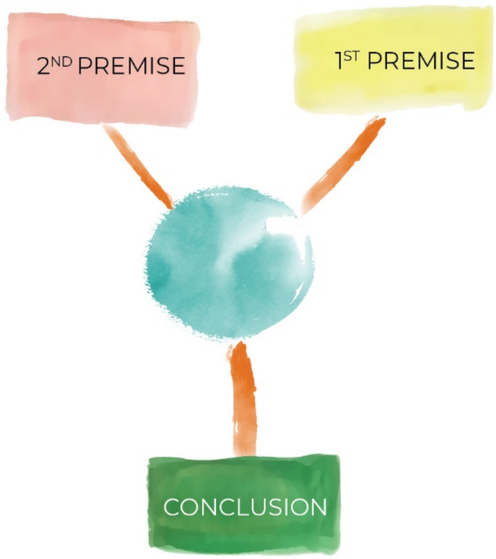
Venn Game. Answer card

Puzzle associated with the card

Briefly, the process to be built is as follows. The second player first places the counters in such a way as to form the two premises on either side, linking the subject and the predicate by the intersection of the two corresponding circles. Then, he moves his counters diagonally towards the centre to bring the two propositions together, just as a bishop in chess moves on a diagonal. Then, such as the knight in chess, he straddles the middle term that links the two premises and makes it disappear to find the conclusion. Finally, by moving his counters vertically such as the rook in chess, he obtains the solution in two circles at the bottom of the game, i.e. in the example here: ‘Some triangles are polygons with three sides and a right angle.’ The construction of the game board must highlight the following symbolic moves that illustrate this deductive reasoning, drawing a conclusion from two premises.

TABLE 5

DIAGRAM OF A DEDUCTIVE REASONING



The major premise (which contains the predicate, noted P):
'All rectangular triangles are (P) polygons with three sides and a right angle' is placed on the right.

The minor premise (which contains the subject, noted S): 'Some triangles (S) are rectangular triangles' is placed on the left.
The group of terms 'rectangular triangles' (noted M) play the role of the middle term in both premises, which here provides a bridge between the two premises.

The conclusion which contains the subject S and predicate P follows below. Clearly, the valid conclusion is: 'Some triangles (S) are polygons with three sides and a right angle' below.

What is written more simply in symbolic form:
AII1 Darii
All M is P
Some S is M
Thus, some S is P

This automated reasoning is well suited to computer programming. A syllogism generator which provides in record time the conclusion of these types of categorical syllogisms can be found in free use on the online site *dcode*[5]. However, the question here is not to use a computer, but simply to move counters on a game board to find a valid conclusion. The same question arises as to whether the construction of the game board is done only with a computer or first by hand. After a few tries with the computer, I concluded that it was better to start by drawing the figures on graph paper using a ruler and compass.

[5] *Syllogisms*. *Generator/Checker/Calculator*. 2020 dCode. [online] Available at: <https://www.dcode.fr/permutations-generator> [Accessed 19 June 2020].

Moreover, making cards and the game board manually can be part of mathematics teaching activities. This gives young children the opportunity to initiate both drawing and mathematical reasoning[6]. For example, with Abbott's novel, they will also be able to establish a link between fiction and geometry[7].

I can recapitulate here the problem of constructing Venn diagram for the Game 5. The issue is: How to build the game board and design the movement of the counters so that the player can easily and visually find the right answer? The answer must be visual and tactile. This is achieved here by first constructing by hand the following game board and playing cards.

[6] Children are invited to learn through drawing several important concepts and theorems: circle, triangle, right-angled triangle, equilateral triangle, inscribed and exinscribed circles, radius, diameter, altitude, median, bisection, perpendicular, bisectors, hypotenuse, line tangent to a circle, parallel lines, symmetry, homothety and translation. These concepts are very useful here to create the game board and playing cards. In particular, they are used to determine the centre of gravity of a triangle manually and without trial and error, using the Pythagorean theorem and the two following geometric properties. First geometric property: The centre of gravity is 2/3 of the median from the top. Second property: In an equilateral triangle, the three medians are also the three altitudes, the three bisection lines, the three bisectors and the three axes of symmetry. Therefore, the centre of gravity of an equilateral triangle is 2/3 of the altitude from the top or 1/3 from the bottom. It is also the centre of the exinscribed circle which passes through the three vertices of the triangle.

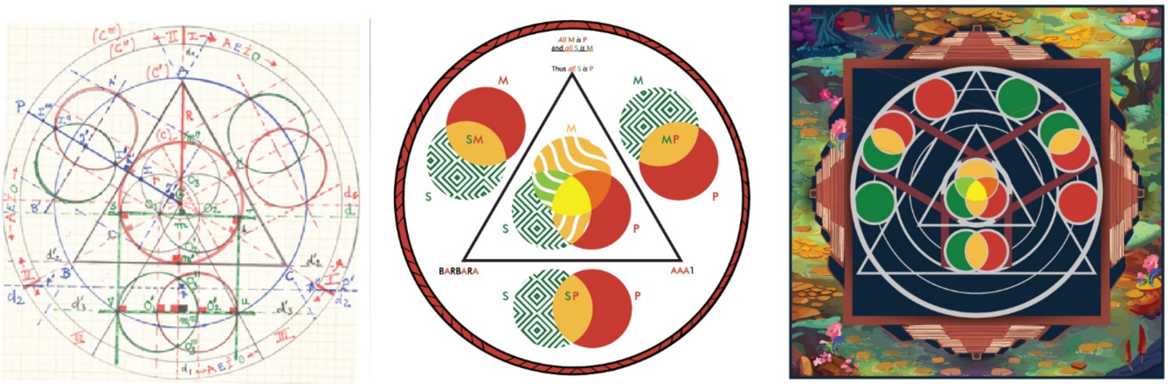
[7] In *Flatland* (1884), Edwin Abbott Abbott brings to life the geometric dimensions, the point, the line, the surfaces, the fourth dimension, thus establishing a link between storytelling and geometry. In the original edition, he chose to illustrate the cover with a pentagon representing the building plan of houses in Flatland.

TABLE 6
PERSONAL ILLUSTRATIONS

SPECIFICATION FOR THE CONSTRUCTION OF THE GAME BOARD AND THE PLAYING CARDS

The ‘computing centre’ is placed inside a first large circle (say C) where the three Venn circles intersect. The premises and the conclusion are represented by three intersecting circles: one circle (O1) for the subject, one circle (O2) for the predicate and one circle (O3) for the middle term which in the premises can be subject or predicate.

Then, the ‘computing centre’ is fed by the outer circles of the premises. The major premise is placed to the right of the first large triangle (say ABC) and the minor premise to the left. By a simple diagonal translation, the two premises intersect in the centre of circle C and, finally, by a vertical translation the conclusion is displayed below the base of triangle ABC. The result is as follows.



Game 5: Construction of the playing cards and the Venn diagrams game board.
Drawing with ruler and compass, then by computer.

The following table summarises the main formulae established for the manual construction of the cards and the game board. Starting from the initial length of a Venn circle (say ‘a cm’), they allow, with or without a computer, to draw the other figures (circles and triangles) at the desired size for the game board and playing cards.

TABLE 7

GAME 5. VENN DIAGRAMS. MAIN FORMULAE FOR THE CONSTRUCTION OF THE GAME BOARD AND PLAYING CARDS

Geometrical figures	Formulae	Application a = 1 cm	Game board a = 3.4 cm	Round playing card a = 1.9 cm
The radius of a Venn circle denoted R_0 = a cm.	R_0 = a	R_0 = 1 cm	R_0 = 3.4 cm	R_0 = 1.9 cm
Centre of gravity (G) of the triangle O_1, O_2, O_3 composed of the centres of the three Venn circles. $O_1G = O_2G = O_3G$.	$O_3G = (\sqrt{3}/3)\ a$ $\sqrt{3} = 1.7321...$ $(\sqrt{3}/3) = 0.5773...$ $O_3G = 0.577a$	O_3G= 0.577 cm	O_3G= 1.9 cm	O_3G= 1.1 cm
Radius of the circumscribed circle (C), denoted r.	$r = (1 + \sqrt{3}/3)\ a$ r = 1.577a	r = 1.58 cm	r = 5.3 cm	r = 3 cm
Altitude of the triangle ABC, denoted H.	$H = (3 + \sqrt{3})\ a$ H = 4.732a	H = 4.73 cm	H = 16	H = 9 cm
Side length of the equilateral triangle ABC, denoted b.	$b = 2\ (1 + \sqrt{3})\ a$ b = 5.464a	b = 5.46 cm	b = 18.5 cm	b = 10.4 cm
Radius of the circle exinscribed to the triangle ABC, denoted R_1.	R_1 = 2r R_1 = $2\ (1 + \sqrt{3}/3)\ a$ R_1 = 3,154a	R_1 = 3.15 cm	R_1 = 10.6 cm	R_1= 6 cm
Radius (R) of the large outer circle of the centre of gravity G.	By design:		R = 20 cm	R = 8.5 cm

To attract attention, especially with regard to children, it seemed interesting to associate the game with the idea of creating a pop up with a large size that would easily captivate the child. The pop-up Game 5 measures 60 cm x 60 cm x 15 cm and the game board 40 cm x 40 cm. The radius (R) of the large circle measures 20 cm for the game board and 8.5 cm for the round playing cards. These circles have the same centre of gravity as the triangle made up of the three centres of the Venn circles (called O_1, O_2, O_3)[8]. At this point the problem arose that I had never built pop ups before, and certainly not of this size. Therefore, as part of this thesis in Cambridge, I learned to make pop-ups mainly from the two reference books by Carter and Diaz (1999) and Finch (2013). Pop-up has today become a paper engineer's job. However, the objective here is not to achieve a technical feat, but to consider the pop-up as a prototype capable of expressing abstract concepts and reasoning. This led me (chapter 2.6 below) to investigate the relationship between the image and the text to be illustrated which I did by studying some fifty works with various supports (albums, pop-ups, games). All the pop-ups presented in this thesis are personal creations, fabrications and illustrations. Two questions remain to be clarified.

1.4 Why choose to illustrate Lewis Carroll's logical works? Why logic?

1.4.1 Why logical tales?

There are at least three main reasons.

1. The first reason is that Lewis Carroll has often been illustrated since John Tenniel's early drawings. So it is only natural for illustrators and me to wish to illustrate it too. However, in my previous studies, although I illustrated well-known traditional tales, I had always considered that it was too early to accept such a challenge. Proposing a reinterpretation of the Carrollian work was risky, especially if I found nothing original to state and illustrate. As Ludwig Wittgenstein writes in the *Tractatus Logico-philosophicus* (1921, 1922, point 7): 'What we cannot speak about we must pass over in silence.' Lewis Carroll's tales are difficult to classify among the usual illustrated stories. The diversity of the literary, poetic and scientific languages he uses in his work are challenging.

[8] Because the centre of gravity of an equilateral triangle is 2/3 of the altitude from the apex or 1/3 from the bottom, the Pythagorean theorem makes it possible to calculate the altitude (h) of this triangle whose base is equal to half the radius (a cm) of a circle of Venn (i.e. a/2 cm) and the hypotenuse is equal by construction to a radius (a cm) of the circle of Venn: $h^2 + (a/2)^2 = a^2$. Hence, $h^2 = a^2 – (a/2)^2 = ¾\, a^2$ and $h = (\sqrt{3}/2)\, a$. Hence, the centre of gravity starting from the apex of the triangle, noted $O_3G = (2/3) \times (\sqrt{3}/2)\, a = (\sqrt{3}/3)\, a$ cm. For the same reason, because the centre of gravity of an equilateral triangle is 2/3 of the altitude from the apex, the altitude of the equilateral triangle ABC can be divided into three thirds, hence $R_1 = 2r$.

However, as the Actors Studio[9] method teaches, whether it is to understand an author or to build a character in the theatre or an illustration, one must first look at the biography of the author and the characters. That is what I was curious to do. The biography is all the more necessary when the author stages himself in his characters. This is the case with Lewis Carroll. He appears in his stories and in his prefaces written for the public[10]. He discusses all the characters of his stories at length with his illustrators: Tenniel, Holiday, Frost, Furniss and Thomson and he exhausted them with his excessive attention to detail (Gattégno, 1974, p. 102). How to illustrate his logical tales without distorting and misinterpreting his thoughts? Who is Lewis Carroll? Gattégno (1974) sheds light in 38 dimensions on the various facets of Lewis Carroll's personality[11]. With Gattégno, one can distinguish three periods (with some unavoidable overlaps): the mathematician, the storyteller, the logician.

TABLE 8

LEWIS CARROLL, MATHEMATICIAN, STORYTELLER AND LOGICIAN

27 January 1832 - 4 July 1862: the mathematician.
Born in Daresbury near Manchester, studied in Rugby and Oxford, professor of mathematics at Christ Church in Oxford (1855–1881), Lewis Carroll published *Euclid's Fifth Book Proven by Algebra* (1858), *Euclid and his Modern Rivals* (1879), *A Tangled Tale* (1883 in a magazine, 1885 in bookstores).

July 4, 1862 – October 1885: the storyteller, the novelist and the pamphleteer.
Lewis Carroll published mainly in 1865, *Alice's Adventures in Wonderland* (started in 1862), *Alice's Adventures Under Ground* (manuscript 1864, published 1885), *Through the Looking-Glass* and *What Alice Found There* (1867–1871), *Sylvie and Bruno* (1873–1889), *The Hunting of the Snark* (1874–1876), *The New Belfry of Christ Church*, Oxford, *The Vision of the Three T'S* (1872–1873), *The Dynamics of a Particle* (1865).

March 1876-14 January 1898: the logician and professor of logic at a secondary school in Oxford (1887). In 1886, Carroll published *The Game of Logic* and in 1896 *Symbolic Logic. Part I. Elementary* (started in 1885).

[9] The Actors Studio is an American membership organisation dedicated to the dramatic arts. It was founded in New York in 1947 by Elia Kazan, Cheryl Crawford and Robert Lewis. His method has become the benchmark in the United States for theatre and film, with the success of former students such as Marlon Brando, James Dean, Robert de Niro, Steve McQueen, Al Pacino, etc. On the method, see compiled by Toby Cole (1955). *Acting. A Handbook of the Stanislavski Method*. Introduction by Lee Strasberg, rev. ed. 1975. New York: Crown publishers, Inc.; Chekhov (1991). *On the Technique of Acting*. New York: Harper Collins Publishers, Inc.; Stanislavski (1989). *Building a Character*. Abingdon-on-Thames, Oxfordshire: Routledge, Taylor & Francis Group.
[10] Carroll is the Dodo in the tales of *Alice's Adventures in Wonderland*; he is 'I' in *Sylvie and Bruno*, his preface is addressed directly to all mothers in *The Nursery 'Alice'* or he warns his readers directly in *Symbolic Logic*.
[11] Gattégno classified them from A to Z, notably: Alice, Stuttering, Girls, Illustrators, Games and invention, Liddell, Mathematics, Photography, Politics, Professorat, Theatre, Victoria, Vivisection, Zeno of Elea.

From this angle, it is easier to notice that the best-known and most illustrated Carrollian works – *Alice's Adventures in Wonderland* – are the work of a person, Charles Lutwidge Dodgson by his real name, who practises several 'languages': that of poetry and tales and that of mathematics. When I discovered he dedicated the end of his life to logic, the idea came to me to read the chronology of his work, beginning at the end and working backwards; that is, by the lesser-known and non-illustrated works, *The Game of Logic* and *Symbolic Logic*. This is a feedback method used in cinema. With this inverted reading grid, it became possible to show that logic is present in most of his tales and that this was rarely highlighted.
What became even more interesting was that a new interpretation could be given to the Carrollian 'nonsense' which in its time was the subject of several controversies.
2. To be able to give an interpretation of the Carrollian nonsense was the second reason for being interested in Lewis Carroll's tales. I was curious to see if it was possible to solve by means of logic and illustration what I call 'the Carrollian paradox of nonsense'. How can it be accepted that Dodgson, the professor of logic, might want to teach children both foolishness in the form of 'nonsense' and things, such as the art of well-thinking, that would be useful to them throughout their lives? A paradox which disappears, as we shall see, as soon as one study the principle of reasoning by the absurd (called *reductio ad absurdum*).
3. The third reason was to take up, now, the challenge by proposing to illustrate logical tales which, to my knowledge, had never been illustrated before, at least for the purpose of illustrating reasoning[12].

1.4.2 Why logic?

I had three main reasons for being interested in Logic I am examining here.

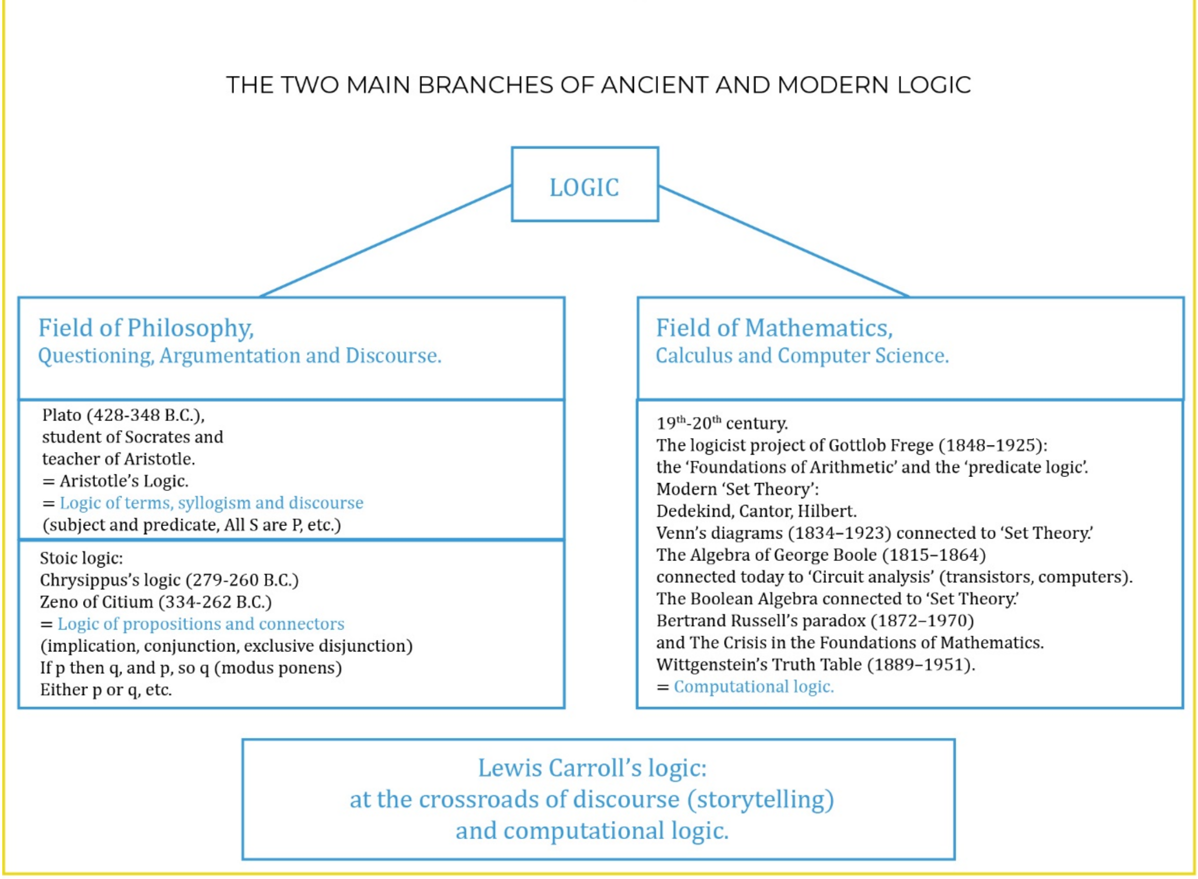
1. The first reason is a challenge for illustrators and myself.

To challenge other illustrators and myself, I raise the following question. Can creative visual arts stimulate children's interest in logic? This discipline, originally a part of philosophy, has become a branch of mathematics. Currently, mathematical logic represents an essential part of computer science. It is interesting to note that Lewis Carroll's logic is situated at the crossroads of the two ever-active branches of logic: on the one hand, questioning, argumentation and discourse, and on the other hand, mathematical calculation and computer science.

[12] Max Ernst's illustrations in the French translation of Lewis Carroll's *Symbolic Logic* (Gattégno and Coumet, 1966, reprint 2006) are admittedly purely decorative and motivational.

What some authors call, for the first, the 'universe of discourse' (or domain of discourse) and, for the second, the universe of calculus, simply called 'Calculus'[13].

TABLE 9



[13] The term 'universe of discourse', which Lewis Carroll uses, is generally attributed to the British mathematician and logician Augustus De Morgan (1806–1871) but the name was also used by George Boole (1815–1864) in his *Laws of Thought* (1854): a mathematical analysis of logic. In his presentation of Boole's logic, Stephen Hawking (2005, ed. 2006, p. 676) writes: 'The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method…' and, p. 706: 'Furthermore, this universe of discourse is in the strictest sense the ultimate *subject* of the discourse.'

While mathematics has an important place in school education, modern and classical logic is little taught at school and generally kept for specialised studies in science. Yet, without paying attention to it, the traditional dialectic – the Art of Thinking – is used in debates as well as in scientific reasoning. Moreover, there is a close link between mathematics and logic, which have foundations in common, such as the principle of non-contradiction. This highlights a fundamental pedagogical question. Should logic be an essential subject to be taught at school? Some will rightly say, fun and entertaining experiences based on observation are enough to stimulate children's thinking and curiosity. This is, for example, and among other common methods, what Maria Montessori's famous Pedagogical Methods (1897) advise parents to do[14]. However, there are areas in science where experiments are not conducted, but conjectures are made. This is another aspect of the challenge for illustrators: the question is no longer how to illustrate observable things but ways of thinking. In terms of reasoning, there is a crucial difference between making 'conjectures' in pure science and conducting 'experiments' in empirical sciences. To discern this, it is possible to refer, for example, to Goldbach's conjecture.

TABLE 10

GOLDBACH'S CONJECTURE

Knowing that a prime number: 3, 5, 7, … is a number which, by definition, can only be divided by itself and by 1, from the number 4 onwards, any even number is effectively the sum of two prime numbers: $6 = 3 + 3$; $8 = 3 + 5$; $10 = 3 + 7 = 5 + 5$, and so on.

[14] The method is practised in schools today in many parts of the world. Based on observation and experimentation in the natural sciences, children observe, for example, the melting of an ice cube in a glass of water to reveal the mystery of Archimedes's Principle. When the ice cube melts, they discover the water level in the glass remains the same. In the same way, children learn how to mix colours. They develop their visual and creative sense of adding colour or, using spotlights, to subtract colours to experience the enigma of light and Newton's prism.

Goldbach's conjecture[15] illustrates the methodological question of reasoning in mathematics. This conjecture has been computer-checked today for integers up to numbers over a billion without ever being faulted. No counter-examples have come to contradict it[16]. Despite some initial demonstrations, to date, no one has ever managed to produce proof. This conjecture will always be true if the calculation is pushed to infinity. This poses the general problem of demonstration and proof in mathematics. In logic, such as in mathematics, a demonstration is consistent if, and only if, it does not contain contradictory statements. Hence the importance of the concept of non-contradiction (highlighted here in Game 4 with the Square of Opposition). Once demonstrated, a theorem remains universally true[17], such as the Pythagorean theorem (demonstrated as a puzzle further on). As Euclid (300 B.C.) has shown, a demonstration is based on reasoning, and since Aristotle, the reasoning is frequently expressed by the theory of the syllogism (Tricot, ed. 1973, p. 46). This raises the question: What is a syllogism and how can it be illustrated? Without being so called nowadays, a syllogism often used in mathematics is that of Euclid's equality syllogism[18]:

$A = B$
$B = C$
Therefore, $A = C$

2. The second reason for studying logic through the visual arts is its pedagogical interest.

If one agrees with Lewis Carroll and others, that logic should be taught as a subject in its own right, then this means making it accessible to young children. This involves teaching them two forms of reasoning. The first, which concerns the 'universe of discourse', is expressed in literary language. The second, similar to mathematics, called 'Calculus', consists of symbolising the first in the form of a logical equation, using a symbolic language specific to logical calculation (allowing the use of Truth Tables, for example, illustrated in Game 7).

---

[15] No one properly knows how the German number theory specialist, Christian Goldbach (1690–1764) discovered that 'any integer greater than 2 can be written as the sum of three prime numbers'. According to the letter he wrote to Leonhard Euler on 7 June 1742, Euler, in his prompt response on 30 June, had reformulated the conjecture in its current canonical form: 'any even number is the sum of two prime numbers'. At the end of his letter, Euler wrote that this was almost certainly correct but had no way of proving it.

[16] This conceptual idea of 'counter-example' indicates it is only necessary to provide a unique example where the theorem is wrong for the entire conjecture to collapse. As no one has discovered a counter-example yet, the enigma of the Goldbach conjecture remains intact.

[17] Among the principles that govern the mathematical universe, there is one that is often overlooked, but which mathematicians respect: mathematical objects are immutable and inalterable (Barthélemy, 2007, p. 8).

[18] Tricot, 1928 (reprinted 1973, 3rd ed., pp. 283–284) shows how to transform a mathematical deduction (A = B and B = C, thus A = C) into a categorical syllogism.

A comparison between the different visual methods of solving syllogisms – known as the diagram method – is made by Lewis Carroll in *Symbolic Logic* (1896) in the Appendix, addressed to teachers. Originally, the Swiss mathematician and physicist Leonhard Euler (1707–1783) had the idea of using drawing to test the validity of Aristotle's categorical syllogisms. To achieve this, he drew circles which translated the abstract principle of the *dictum de omni-dictum de nullo*[19] into a concrete image. The following table gives an example of a model of reasoning, in its literary and symbolic form, to which tradition has given the name Barbara (by inverting the order of the premises). The Euler diagram visually proves that the conclusion is valid, not because it is known that 'all dogs are animals', but rather because the conclusion is entirely inferred from the two premises.

TABLE 11

EXAMPLES OF WHAT TRADITION CALLS A CATEGORICAL SYLLOGISM

Forms	Standard form	Literary form	Syllogism form
Premise 1	X is Y	All dogs are mammals,	All S is M
Premise 2	Y is Z	and all mammals are animals	All M is P
Conclusion	Thus, X is Z	Thus, all dogs are animals	Thus, all S is P

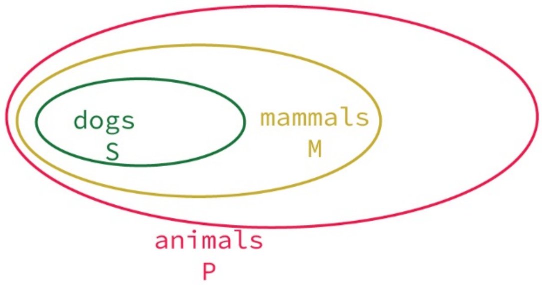
From the point of view of illustration, to represent a categorical syllogism into an image, it is not enough to illustrate the subject (S: dogs) and the predicate (P: animals)), because the main objective is to show the conclusion is deduced from the two premises via a middle term (M: mammals). While Aristotle in the *Organon* literarily defines rules to establish the validity of a syllogism, the idea of Euler, Venn, then Lewis Carroll uses drawing to prove the validity of the conclusion. Here is an example of this image/text ratio that highlights reasoning.

[19] Victor Thibaudeau (2006, p. 720) sums up the *dictum* as follows (I translate into English): When a term is universally assigned to a subject, it must necessarily be assigned to everything contained in the extension of that subject (*dictum de omni*). When it is denied, it must necessarily be denied (*dictum de nullo*) to that contained within the extension.

TABLE 12

EULER'S CIRCLES

1. All dogs are mammals
2. All mammals are animals
3. Thus, all dogs are animals



Euler's circles (ellipses here) illustrate the abstract and complex principle of the *dictum de omni*: the term dogs is contained in the term mammals, and the term mammals is itself contained, in the term animals. Therefore, the term 'dogs' is *necessarily* contained, in the term animals. This syllogism is valid because 'the conclusion follows from the premises'[20]. However, unlike geometry, these circles – which can be ellipses as used by Peter Kreeft (2004) – possess no geometric value. They serve to delimit spaces, or, in other words, to classify things in boxes. This representation was taken up by Venn (1881) and later in the set theory of the German mathematician Georg Cantor at the end of the 19th century. Even if Euler's circles provide no geometric utility, it remains an interesting idea for an illustrator to use the image/text ratio in this way to introduce the notion of reasoning and demonstration. Moreover, it is interesting to see that the representation of a syllogism does not express an opinion or a personal interpretation. The conclusion comes from a demonstration which is based on precise rules, which are independent of the one who expresses the reasoning. As is shown below, it is the position of the middle term and its definition in both premises that makes it possible to judge whether the reasoning is valid or fallacious. The middle term position allows to establish four valid models of reasoning, called Figures, which I illustrate in Games 1 to 3 with playing cards and puzzles.

Figure I	Figure II	Figure III	Figure IV
M – Predicate	Predicate – M	M – Predicate	Predicate – M
Subject – M	Subject – M	M - Subject	M – Subject

[20] It can be observed that all the terms of the conclusion (subject 'dogs' and predicate 'animals') are entirely contained in the premises. This is a major difference with the Hegelian model: thesis, antithesis, synthesis and models of the 'advantage, disadvantage, synthesis' type where the elements of the synthesis are not entirely contained in the premises, but are 'something else' as shown by Ellul (2003). This avoids a frequently unanswered question: where does 'that something else' come from?

Considering that it is important to teach children logic, Lewis Carroll wrote for them *Game of Logic* and *Symbolic logic*, two books that provide a large number of examples of the resolution of categorical syllogisms belonging to the four Figures I,II, III, IV. His method of diagrams uses squares and not Euler or Venn circles and employs a special kind of arithmetical calculation[21]. It is this method that I illustrate in Game 6 using a pop-up, a game board, figurines, counters and playing cards. In Booklet 6, such as Lewis Carroll does in the manual that is *Symbolic Logic*, I first present how to translate categorical syllogisms into what he calls 'bi and triliteral' diagrams. Concretely, consider the following example of that he chose for children.

[21] As Kreeft (2004, ed. 2014, p. 237) writes, the visual method of Euler's circles 'will not give a clear result for some syllogisms with I or O premises (perhaps 5–10% of the syllogisms you will meet.' What made the British mathematician and logician John Venn (1834–1923) well-known, following the work of Leonhard Euler, was his method to visually solve Aristotelian syllogisms. In *Symbolic Logic*, Lewis Carroll writes (Dover ed. 2015, pp. 174): 'Mr. Venn's Method of diagrams is a great advance on the above Method' (Euler's Method), however, Carroll writes, p. 176: "My Method of Diagrams *resembles* Mr. Venn's, in having separate Compartments assigned to the various Classes, and in marking these Compartments as *occupied* or as *empty*; but it *differs* from this Method, in assigning a *closed* area to the *Universe of Discourse* …' In other words, everything takes place inside the squares, contrary to the Venn method.

TABLE 13

AN EXAMPLE OF LEWIS CARROLL'S SYLLOGISM RESOLUTION

All diligent students are successful
All ignorant students are unsuccessful

Conclusion:
All diligent students are learned, and all ignorant students are idle
('not ignorant, i.e. learned' and 'not diligent, i.e. idle').

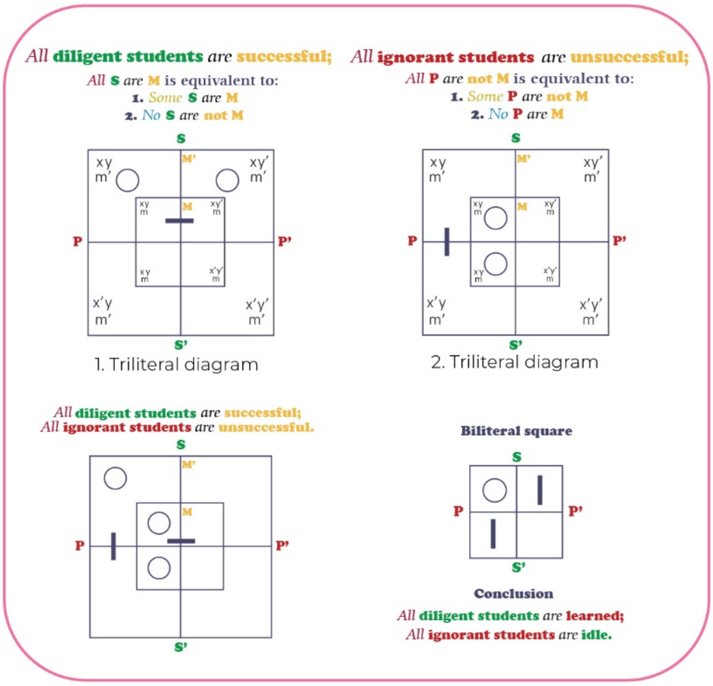
Lewis Carroll, ed. Dover, 2015, p. 64.

Demonstration:
Let the 'Universe of discourse' be 'Students'; m = successful students; x = diligent students; y = ignorant students

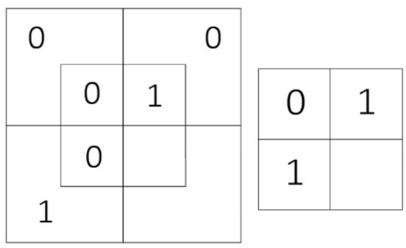
Coded Carrollian syllogism:
All x are m
All y are m'

Conclusion: All x are y'; and All y are x'
With the symbolisation showing the subject (S), the predicate (P) and the middle term (M):
All S are M
All P are not-M

Conclusion: All S are not-P; and All P are not-S
Carrollian Interpretation of text in image

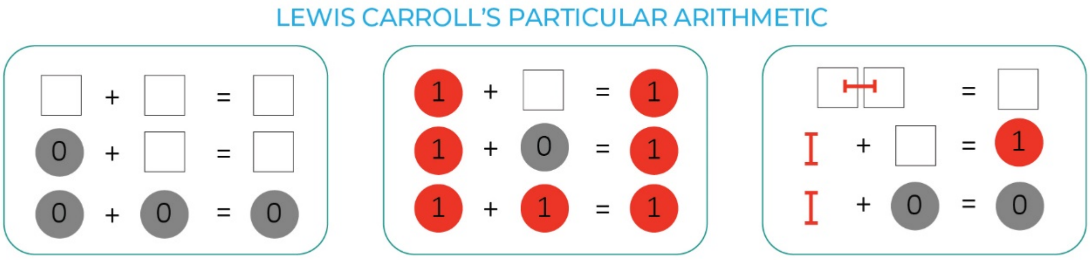


The *Game of Logic* and *Symbolic Logic*, explain how to group the two premises into a single triliteral diagram, and then how to switch from this to a biliteral diagram to obtain the valid conclusion of the syllogism. The valid conclusion displayed on Lewis Carroll's biliteral diagram is presented in its following minimalist visual version:



Triliteral and bilateral carrollian diagrams used to visually find the conclusion of a syllogism (Lewis Carroll, 2015, p. 62)

As Anne d'Alleva[22]points out, the interpretation of the images, whether it is a work of art, or as here Euler, Venn's circles, Lewis Carroll's squares or a logical puzzle, is not a matter of personal opinion. The painter, as well as the logician here, created the image (the painting, the drawing of circles and squares) within the framework of an academic discipline which, depending on the period, has its methods, its rules, its customs, its ancient or modern concepts. For illustration, it is therefore not a question of describing the images (the intersection of the three circles of Venn, the superimposition of two 'bi- and triliteral' squares by Carroll), but of analysing them, here in a formal way. The analysis of the image/text ratio expresses reasoning here. In Game 6, it is this mode of reasoning that is to be understood through play, by means of the visual arts. This analysis allows me to highlight the original way in which Lewis Carroll uses arithmetic to solve literary syllogisms.



The reconstitution of the Carrollian logic puzzle with regard to the history of logic allows to better understand the originality of its particular arithmetic. Like a special addition table, this arithmetic illustrated here makes it possible to find algebraically the conclusion of a literary syllogism.

This Lewis Carroll method announces the Boolean binary logic of 0 and 1, used in computer science. It allows a literary text (a syllogism), i.e. a short story or argument, to be translated into the form of a logical equation, whose reasoning and conclusion are verified by calculation. This formal analysis of the image/text ratio allows me in Game 7 to establish in the form of a pop-up game a link between storytelling and logic to determine the validity of reasoning, using Truth Tables and Natural deduction. As will be seen, syllogism theory makes it possible to distinguish truth from premises and the validity of reasoning. This leaves the authors of tales and stories a great deal of freedom and creativity in the choice of words and sentences. Consequently, I will return to this important point later, there is no 'nonsense' in the specimen of the following syllogism proposed by Lewis Carroll.

[22] Alleva, A.d'., 2010. *How to Write Art History*. Reprint 2019, 2nd edition. London: Laurence King publishing, pp. 74–76: Art historical arguments: opinion vs. interpretation.

TABLE 14

TRUTH AND VALIDITY

Lewis Carroll syllogism:
'All cats understand French;
Some chickens are cats'.
Therefore, 'Some chickens are creatures understanding French.'

A formal analysis of this syllogism shows that it is valid. It is even a perfect figure of a syllogism which even bears Darii's name to specify that the first premise is a universal and affirmative proposition (A-form), the second premise and the conclusion are particular and universal propositions (I-form).

Darii syllogism:
All M is P;
Some S is M.
Therefore, some S is P.

Valid syllogism from Lewis Carroll's *Symbolic Logic* (ed. 2015, p. 57)

On a pedagogical level, logic allows children to understand that there are valid and fallacious reasonings, such as this amusing example of Ionesco. A candid conversation, as the one between the Logician and the Old Gentleman in Ionesco's play (*Rhinoceros*, act I, 1959), shows how easy it is to deduce incorrect conclusions in a discourse composed of only three sentences: two premises and one conclusion.

TABLE 15

IONESCO'S FANTASY SYLLOGISM OF THE CAT AND THE DOG

'Logician [to the Old Gentleman]: Here is an example of a syllogism.

The cat has four paws.
Isidore and Fricot both have four paws.
Therefore, Isidore and Fricot are cats.

Old Gentleman [to the Logician]:
My dog has got four paws.

Logician [to the Old Gentleman]:
Then it's a cat.

Old Gentleman [to the Logician, after deep reflection]:
So then logically speaking, my dog must be a cat?

Logician [to the Old Gentleman]:
Logically, yes. But the contrary is also true.'

(Rhinoceros, act I, 1959)

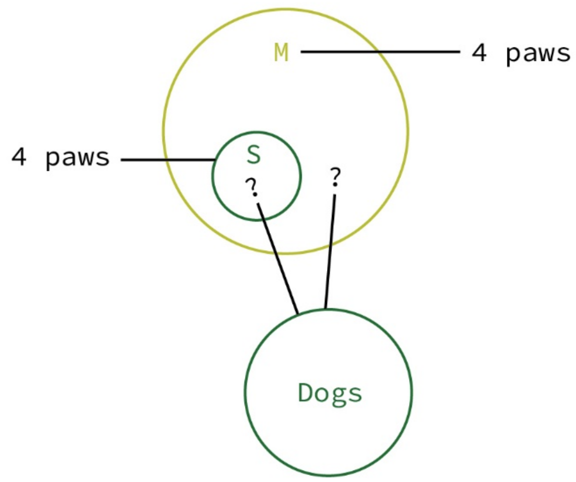
Behind the nonsense of Ionesco's syllogism lies, in a few lines, the Aristotelian theory of valid and invalid reasoning, with the problem of opposites (dog, non-dog), the inversion of terms (subject, predicate) in a sentence, the position of the middle term and, here, a misuse of the *dictum de omni* principle in the middle-term theory. As Kaye (2009, ed. 2017, pp. 54–55) does, one can use Euler's circles to show here the invalidity of Ionesco's reasoning. The misuse of the middle term even has a name: 'undistributed middle'. Because not all cats and dogs are the only animals to have four paws, we do not know in which circle to inscribe the term 'dogs'.

TABLE 16

USING THE EULER DIAGRAM TO DENOUNCE FALLACIOUS REASONING

All cats (P) have four paws (M)
All dogs (S?) have four paws (M)
All dogs (S) are cats (P)?

Invalid syllogism.



The middle term fails here to bridge the gap between the premises and the conclusion before disappearing. The coding of the simple words cat, dog, four paws by letters S, M, P, the term 'undistributed middle' and all these notions together might seem too abstract and complex to understand for young children, and ultimately not very entertaining. However, if the intention is not to deprive them of the finesse behind the nonsense of Ionesco's syllogism, it may be useful to make them understand why this reasoning is incorrect. This will allow children to subsequently perceive one of the key rules of the syllogism: to go from the premises to the conclusion, the middle term must be universal (distributed) once at least in the premises. This syllogism is not valid here, not because it is elementary to observe that a dog is not a cat, but because the reasoning is incorrect. Consequently, any reasoning in the following form is invalid.

TABLE 17

INVALID FORM OF REASONING

All P is some M	All cats are some four-paw animals
All S is some M	All dogs are some four-paw animals
Therefore, invalid conclusion:	**Invalid conclusion:**
All S is All P	All dogs are all cats

By replacing letters S, M, P with words, this syllogism makes it possible to create as many stories of nonsense as one wishes. However, depending on the age of the children, the application of Euler, Venn and Lewis Carroll diagrams can be more or less difficult to understand. It requires the reading of an instruction manual. This is the purpose of the illustrated booklets I have designed to be used with the games. Nevertheless, having been confronted myself with the problem of understanding and illustrating Venn and Lewis Carroll's diagrams, I finally found a simpler way to introduce children to the Art of thinking. This way uses a principle that children are familiar with: the construction of puzzles. Among the 40 puzzles I have made, the following one is an interpretation of the nonsense of Ionesco's Logician.

PERSONAL ILLUSTRATIONS

ILLUSTRATION OF IONESCO'S SYLLOGISM OF THE CAT AND THE DOG
EXAMPLE OF AN INVALID SYLLOGISM

In this puzzle, the sign  makes children aware that not all reasoning and conclusion derived from arguments (premises) are necessarily correct (valid). They will observe that the middle term, in yellow on the jigsaw puzzle, is placed at the extremes and that there are other animals represented than cats and dogs. The middle-term ‘paws’ fails to link in the conclusion the subject ‘dogs’ and the predicate ‘cats’. The reason is the term ‘paw’ is not a general characteristic of all animals: ‘all the cats and all the dogs’ do not represent ‘all the animals’ as evidenced by the other animals portrayed on the puzzle. By comparing the different valid and invalid puzzles and by consulting the Booklets, with the help of adults, they will be able to establish the difference between a general case and a particular case, i.e. what is universal: ‘all dogs’ and what is particular: ‘having four paws’.

From an educational standpoint, logic distinguishes between two expressions: ‘To give an opinion’ and ‘to give an argument’. In logic, Antony Weston explains (1954, 5th ed. 2017, p. xiii): ‘*To give an argument* means to offer a set of reasons or evidence in support of a conclusion.’ Among the forms of argument, tradition generally distinguishes Aristotle’s categorical syllogisms from the compound syllogisms of the Stoics, which use several symbols, signs and logical connectors (and, or, implies, etc.). Some models are commonly used without it being known that they are, for example, *the modus ponens* defined by the Stoics. Nevertheless, these same models can be misused, which causes fallacious reasoning to which tradition has given names. For example, it is a mistake called ‘fallacy of affirming the consequent’ to confuse the material implication (If … then) with the word ‘therefore’ which indicates the valid conclusion of reasoning[23]. The games designed in this thesis allow children to become familiar with different models from many examples given in the Booklets.

[23] As Barthélemy writes (2007, p. 49, I translate into English): ‘it is a serious mistake to translate (p => q) by “p is true, so q is true”. On the other hand, if one uses the form “p implies q”; and p is true; therefore, q is true’, the reasoning is correct. As seen below, it is the correct use of the *modus ponens,* which allows reaching a valid conclusion. In Lewis Carroll’s *What the Tortoise Said to Achilles* (1894), Achilles ignores this warning. This will lock him into a paradox from which he will not be able to escape.

TABLE 18

EXAMPLE OF DIFFERENT FORMS OF SYLLOGISMS AND SYMBOLS

Aristotelian form	Modus ponens stoic syllogism	Mathematical deduction of a poly syllogism	Logical equation of a dilemma
Lewis Carroll: Literary Syllogism: All diligent students are successful All ignorant students are unsuccessful **Conclusion:** All diligent students are learned, and all ignorant students are idle (Lewis Carroll, ed. Dover, 2015, p. 64)	**Chrysippus:** Literary Syllogism: If the first, then the second The first **Conclusion**: The second	**Euclide:**	**Corax's dilemma:** Tisias either wins his trial (p) or loses his trial (r = not-p), i.e. p or r, coded: p + r If he wins, according to the contract he has to pay (q), coded: p => q If he loses his trial, he has to pay (q), coded: r => q In either case, he has to pay (q) **Conclusion:** q
Symbolisation of the syllogism All x are m All y are m' **Conclusion:** All x are y'; and All y are x'	**Symbolisation** If p then q And p **Conclusion:** q	A = B B = C **Conclusion:** A = C	This 'simple constructive dilemma' is written: [(p + r) . ((if p => q) . (if r=> q))] ∴ q

3. A third reason to be interested in logic is the possibility of reinterpreting the carrollian nonsense.

In Lewis Carroll's work, 'nonsense' can take several forms. It may be the strange conclusion of a reasoning that claims to be valid: 'Some chickens are creatures understanding French," or the reasoning in the form of Ionesco's syllogism: 'All dogs are cats,' or combinations of letters whose words have no meaning, such as in Borges' Library combining all the letters of the alphabet[24]. Therefore, I examine three aspects of nonsense.
Firstly, puns raise an important issue here: Can everything be taught through the play? To distract children, Lewis Carroll had invented a significant number of word games: *word links*, *doublets*, *lanrick*, *mischmasch*, *syzygies,* jabberwocky (*gattégno, 1974,* pp. 107–117). They are based on the idea of substituting a letter in a word to bring up an alternative word, or to constitute words from two or three letters proposed by the opponent, etc. It is not surprising that in the syllogisms, he imagines sentences where the subject and the predicate are inverted, and where the negation is not about the verb but on the subject or the predicate. The problem here is that these inversions follow precise logical rules (conversion, obversion, partial and full contraposition). When one knows the rules of inversion, the question of nonsense no longer arises. What may appear to be nonsense is in reality equivalent logical propositions, even if they are unusual. In Games II and III, called The Mirror Game, I employed puzzles to illustrate the logical mechanisms of transformation, sometimes humorous, of one proposition into another logically equivalent one. That is to say if the first proposition is true, the second is also true. For example, the proposition: 'All human beings are mortal' is logically equivalent to 'All non-mortals are non-human', which can create a pun (non-human = inhuman person, without pity, without generosity). Puzzles allow to visually establish conversions.

[24] Borges, J. L.,1956. *Ficciónes*. Translated from Spanish by A. Hurley, 2000. *Fiction*s. London: Penguin Books, modern classics, pp. 65–74.

PERSONAL ILLUSTRATIONS
GAME III. THE MIRROR GAME II

OBJECTIVES

Through the game of inversion (conversion, obversion, contraposition), it is fortunately possible to find with the illustrated puzzles the same conclusion as the syllogisms produced by the Venn and Lewis Carroll diagrams.

An illustrated example of full contraposition:
Hence the formulae (Hurley, 2011, p. 210):
first obvert, then convert, then obvert again,
i.e. switching the subject and predicate and exchanging them for their term complements (non-S, non-P).

A- Form: All human beings are mortal.

1. obverse – > eq. E-Form: No human being is non-mortal.

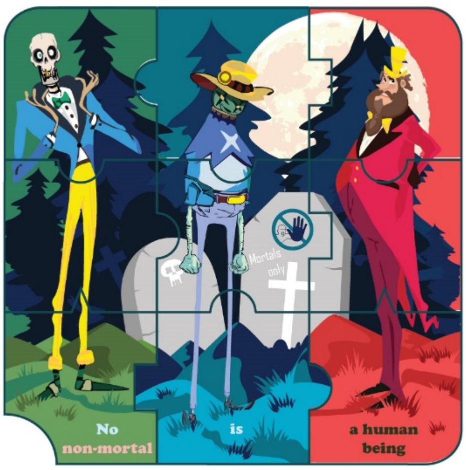
2. converse – > eq. E-Form: No non-mortal is a human being.

3. obverse – > eq. A-Form: All non-mortals are non-human.

So, A-Form: All human beings are mortal

Is equivalent to: A-Form: All non-mortals are non-human.

Note: In this example, after the conversion, we also invert in the puzzle the colours to keep the subject's green and the predicate's red.



Secondly, another interpretation of nonsense has been known since antiquity and found in the '*reductio ad absurdum'*. Reasoning by the absurd is paradoxically based on the principle of contradiction applied to nonsense. This will lead me to illustrate in game 7 with logical formulae, the paradoxical conjunction between nonsense and contradiction.
Thirdly, the paradoxes that Lewis Carroll draws on writing *'Barbershop Paradox* titled *'A Logical Paradox'* (1894) and *'What the Tortoise Said to Achilles'* (1894) also use nonsense to re-establish truths. This will lead me to distinguish between true, false and antinomic paradoxes.
In summary, beyond the defenders of nonsense such as Chesterton[25], the concept of nonsense assimilated to fantasy has a rational utility. As paradoxical as it may seem, it can help to reason correctly. It is this interpretation that I propose to illustrate. Moreover, the Carrollian nonsense brings us back to the author, Charles L. Dodgson, logician and mathematician.

TABLE 19

A DEFENCE OF NONSENSE: THE DEFENDANT BY G. K. CHESTERTON

For Chesterton, Edward Lear with his nonsense rhymes is both chronologically and essentially the father of nonsense . However, he also mentions Lewis Carroll:

Lewis Carroll: 'his strange double life in earth and in dreamland emphasises the idea that lies at the back of nonsense – the idea of escape, of escape into a world where things are not fixed horribly in an eternal appropriateness, where apples grow on pear-trees, and any odd man you meet may have three legs'.

'Lewis Carroll's Wonderland is purely intellectual.'

'Carroll works by the pure reason.'

'Carroll, with a sense of mathematical neatness, makes his whole poem a mosaic of new and mysterious words.'

Excerpt from Chesterton (1901)

1.4.3 To delimit the subject of research: Logic as a tool[26]

This overview of logic helps to delimit the subject of study. The whole point of logic, used as a tool, is to distinguish correct reasoning from fallacious one. Ultimately, it highlights the question: how can the visual arts contribute to helping children reason correctly?

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[25] In *The annotated Alice*, Martin Gardner (1960, ed. 2001, p. 327) gives references to authors who have addressed the question of Nonsense. Among them is: Chesterton, G.K., 1901, A Defense of Nonsense, The defendant. London: *The Daily News*. Reprint 2004. The Project Guttenberg (2004), Ebook of The Defendant, by G.K. Chesterton. [online] Available at: <https://www.gutenberg.org/files/12245/12245-h/12245-h.htm>. [Accessed 9/10/20].
[26] '*Logic as a tool. A guide to formal logical reasoning*' is the title of Goranko's book (2016).

Logic can be applied at the level of reasoning and thinking to many disciplines: mathematics, philosophy, literature, etc. However, it is not a question here of philosophy, rhetoric, psychology, neuroscience, non-standard logic or language studies. In particular, the term metalanguage (from the Greek μετά-, Meta, meaning 'after' or 'beyond') is used in this thesis, for both logic and illustration, to distinguish between the everyday practice of these languages, on the one hand, and the axioms and rules or 'grammar' of these languages on the other[27]. It is only a question of enabling young children and older students to employ logic through the visual arts as a tool, i.e. in the literal sense of the definition of Aristotle's *Organon*, the word meaning Tool or Instrument in ancient Greek (ὄργανον). The issue of establishing conjunction between visual arts and discipline as complex, formal and abstract as logic already seems to me a big enough challenge to take up, as Lewis Carroll found out when he drew its 'bi and-triliteral' squares designed to help children understand the logic or the Art of thinking[28]. Seeking first to understand what logic is before trying to illustrate it raises the prior academic question of bibliographical research. The method followed here consisted of referring to relatively elaborate works in the field of logic, mainly at university level, and cross-referencing the different points of view given by their authors in order to retain in the Booklets only what is generally known to professionals and which, in science, is not disputed.

1.5 Bibliographical research

Concerning Lewis Carroll, there are three main and original materials: his Diary[29], the illustrated tales, including the three versions of Alice that can be compared, the memoir of his nephew Collingwood, letters or memoirs of friends and illustrators close to the author. In addition, there are the Carrollian logical works: *The Game of Logic*, *Symbolic Logic*, as well as *A Tangled Tale*, writing on Euclid and paradoxes, and many writings and theses on Lewis Carroll and his work. I refer in particular to the writing and doctoral thesis of Jean Gattégno who promoted Carroll at La Sorbonne in Paris in 1970. This is for several reasons. First, he is one of the few authors interested in Lewis Carroll's complete body of work, both in literary works and in logical and mathematical works.

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[27] Lerot, J., 1993. *Précis de linguistique générale*. Paris : Les éditions de minuit, p. 24 (I translate into English) : « The scientific discourse used to describe the structure and functioning of a natural language is a metalanguage. Grammar rules and grammatical terminology are therefore metalanguage ».

[28] 'It has cost me years of hard work', writes Lewis Carroll in his preface to the fourth edition of *Symbolic Logic* (1896).

[29] Lewis Carroll kept a diary that was originally only accessible to his nephew Stuart Dodgson Collingwood. The latter had published some excerpts after the author's death in 1898, including *Life and Letters* of Lewis Carroll and another book, *Lewis Carroll, Photographer*, published in 1949, also contained excerpts. It was only in 1984 that Dodgson's nieces decided to publish *The Diaries* for both researchers and the general public. (*The Diaries of Lewis Carroll*. The Executors of the Estate of the Late C. L. Dodgson and the Late Roger Lancelyn Green. Translated by P. Blanchard and J.-P. Richard and annotated by J. Gattégno, 1990. *Journal.* Paris: la Pléiade, Gallimard.)

He coordinated the editing and translation of Lewis Carroll's Works in the prestigious collection of La Pléiade (Gallimard, 1990). Secondly, translation problems from English to French remain a concern especially when an author like Lewis Carroll plays with words and language. It is a problem comparable to the one the illustrator faces when moving from words to drawing or from drawings to words.

To reconstruct the paths that the mathematician and logician Charles Lutwidge Dodgson most likely followed through the authors he cites: Euler, John Venn (1881) and his discussions with Oxford logicians, I refer to twelve main works (supplemented by other works and tutorials broadcast on the online YouTube site cited in the appendix bibliography).

The titles of the works cited give an idea of the subjects illustrated in the thesis. This main bibliography, which concerns logic, is supplemented in the appendix by bibliographical references relating to illustrations and illustrators, pop-ups, games and the calculation of statistical ratios.

Within the framework of the thesis and the Game Booklets, the bibliographical research consisted of referring to fundamental works appropriate for university study into the teaching of logic. Here, my aim is not to write a treatise on logic from these works, but to make the basic principles they contain more accessible through the visual arts. This led me to cross-reference the different insights given by their authors, firstly to make sure that I understood the principles before illustrating them, and secondly to retain only what is generally accepted by logicians to be correct.

TABLE 20

MAIN BIBLIOGRAPHY ON LOGIC (COMPLETED IN THE APPENDIX)

Aristotle, 384-322 BC. *Organon.* Reprint and translated by J. Barnes, 1998; Reprint and translated and introduction by C.D.C. Reeve, edited by R. McKeon, 2001. *The basic works of Aristotle. Organon:* Categoriae, p. 7, De Interpretatione, p. 40, Prior Analytics, p. 65, Analytica Posteriora, p. 110, Topica, p. 188, On sophistical refutations, p. 208. New York: Modern Library Paperback Edition. Reprint and translated into French by J. Tricot, 2008–2012. Paris: J. Vrin.

Chenique, F., 1975. *Éléments de logique classique. L'art de penser, de juger et de raisonner.* Paris: Dunod-Bordas. Reprint 2006. Paris: L'Harmattan.

Goranko, V., 2016. *Logic as a tool. A guide to formal logical reasoning.* Chichester, West Sussex: John Wiley & Sons Ltd.

Hurley, P. J., 2005. *A concise introduction to logic.* Reprint 2008, 10th ed. London: Thomson Wadsworth.

Kreeft, P., 2004. *Socratic logic. A logic text using Socratic method, Platonic questions, and Aristotelian principles.* Reprint 2014, Edition 3.1. Indiana: St Augustine's Press.

Lee, S.-F, 2017. *Logic. A complete introduction.* London: Hodder and Stoughton.

Parsons, T., 2014, *Articulating Medieval Logic.* Oxford: Oxford University Press.

Quine, W.V., 1976. *The ways of paradox and other essays.* Cambridge, Mass: Harvard University Press. Translated into French by S. Bozon and S. Plaud, 2011. Les voies du paradoxe et autres essais. Paris: J. Vrin.

Thibaudeau, V., 2006. *Principe de logique, définition, énonciation, raisonnement.* Laval, Québec: Presse de l'Université de Laval.

Tricot, J., 1928. *Traité de logique formelle.* Reprint 1973, 3rd ed., Paris: J. Vrin.

Watson, J.C. and R. Arp, 2015. Critical thinking. *An introduction to reasoning well,* 2nd ed. London, Oxford: Bloomsbury publishing Plc, in particular, chap. 5, Truth Tables, pp. 141–167, chap. 6: Rules of deductive inference (modus ponens, modus tollens), pp. 168–209.

Wittgenstein L., 1922. *Tractatus Logico-Philosophicus.* Translated from German by D.F. Pears and B. F. McGuinness. Introduction by B. Russell, 2014. London and New York: Routledge Great Minds. Translated into French and notes by G.-G. Granger, 1993. Paris: Gallimard.

Conclusion of chapter I

To reinterpret Carroll's nonsense, there was no other solution than to retrace the path that Lewis Carroll had certainly had to follow from Aristotle and the Stoics to the modern logic that was beginning to take shape. How could one interpret and illustrate a text that one does not understand? This made me aware of the importance of the work of the Cambridge logicians at the turn of the 20th century, notably with Bertrand Russell's *Principia Mathematica* (1910)[30] and the Truth Tables of Wittgenstein's *Tractatus Logico-philosophicus* (1921). It became almost a moral obligation for me to take a serious interest in logic in a thesis presented at Cambridge where there were such eminent logicians. As the Booklets show, it was not possible to skim over the subject. It had to be studied conscientiously. However, this is not a thesis on logic. All the concepts and reasoning presented here are widely known to professionals in the field as shown by the works consulted. The study of logic[31] was taken as far as my understanding allowed to determine the extent to which the language of illustration, or in a broader sense, that of visual arts, could help illustrate scientific languages and participate in discoveries in this field. Thus, the aim here was not to write a treatise on logic, but to be able to illustrate its main universal principles.
Chapter II consists of constructing and deconstructing what I call the 'Carrollian logic puzzle', that is, the path that Lewis Carroll probably followed from the old logic to the modern logic. Then, I research methods to illustrate abstract concepts and I study the image/text ratio to determine how to illustrate a text whose main objective is to highlight reasoning. I end this chapter II with a case study of about fifty illustrated books in different media: digital, albums for children, pop-ups and games to draw lessons for my illustrations concerning the Art of thinking.

[30] The *Principia Mathematica* (1910–1913) is a three-volume work on the foundations of mathematics written by Alfred North Whitehead and Bertrand Russell of Trinity College, Cambridge.
[31] The study is limited to the logic of terms and propositions, the Boolean logic of Truth Tables and the beginning of the Predicate Calculation (Chauve, 2015, pp. 23-40.). It does not consider non-standard alternative logic, such as fuzzy logic.

Chapter II

Using visual Art to help children understand the Art of thinking

At several times, Lewis Carroll adopts the language of pure and formal sciences in his tales. If not 'decoded', it may seem incomprehensible to most people. It raises two crucial issues. The first problem consists in discovering in his tales the implicit logical reasoning that often lies behind the fiction, nonsense and fairy tales. This is what I call the question of interpreting from the point of view of logic. Once discovered, the second problem is how to employ visual arts to help children discover the logic in an explicit, entertaining and useful way. This is the problem of illustration in the broadest sense of visual art. Is the language of visual Art elaborate enough to understand and illustrate, amongst other subjects, the language of the pure sciences: arithmetic, geometry, algebraic equation, logic? If visual art can help to solve the two problems raised here, then what might Carroll's nonsense, paradoxes and dilemmas reveal to us? So, this chapter begins with an illustrative interpretation of selected texts from Lewis Carroll's work.

2.1 Three models of reasoning from Lewis Carroll's texts

Learning to argue is probably one of the most complex and abstract learning experiences for young children. The process requires a minimum of knowledge or creativity to establish arguments (premises) and the mastery of several logical operations to deduce valid conclusions. These operations concern, for example, the relations of contradiction between arguments, the ability to infer consequences from causes, the choice of hypotheses and logical connectors linking the premises to the conclusion. The analysis of the following three texts by Lewis Carroll allows understanding into how argumentative reasoning is constructed, the aim being to illustrate it.

2.1.1 First model: The *modus ponens*

Supporting text by Lewis Carroll (1864). *Alice's Adventures Under Ground*[32].

1. Analytical interpretation of an excerpt from Lewis Carroll's text.

The aim of this interpretation, and its illustration, is to render the reasoning explicit, even for young children.

[32] Carroll, L. 1864. *Alice's Adventures Under Ground.* Original manuscripts illustrated by Lewis Carroll. Reprint 2019. London: The British Library, pp. 7–8, 11, 60–62.

The text *of Alice's Adventures Under Ground*, imagined by Carroll during a legendary boat ride[33] on July 4, 1862, was initially written for Alice Liddell, the ten-year-old daughter of the Dean of Christ Church, Oxford, with no intention of publication.
Let us consider the three texts composed in the form of a syllogism.

TABLE 21

'Drink Me', you'll shrink;
Alice drinks the potion.
(considering that the bottle does not contain poison; which was then a hypothesis).
So she shrinks ('I must be shutting up like a telescope', pp. 7–8).

It is a fairy tale way of presenting a known reasoning, called the *modus ponens.* Its foundations can already be found in the Stoic philosopher Chrysippus of Soli (around 280-207 BC). It was symbolised in the Middle Ages in its current form:

Chrysippus of Soli	Modus ponens
If the first, the second	If P then Q
The first	And P
Therefore, the second	Therefore, Q

2. Preliminary pop-up: The Logical Spring

What was only a hypothetical syllogism at the beginning of the tale (Alice wonders if it is poison or not), becomes a categorical syllogism. Each time she uses the process, she shrinks. Which makes it a comic spring in fairy tales, a 'logical' spring.
In sum, the 'logical' spring highlights a mechanism of thought. It is a game that children play at junior school, without being aware that this is an illustration of a logical reasoning known since antiquity. It is this mechanism that I illustrate in a pop-up called for this reason The Logical Spring (Preliminary Game 1).

[33] It is a boat expedition on the river at Gostow, with Lewis Carroll, his friend Duckworth and the three little Liddell girls, including Alice. (Gattégno, 1974, p. 28.)

PERSONAL ILLUSTRATIONS

THE LOGICAL SPRING

Chrysippus of Soles
If the frist, the second
The first
Therefore, the second

Modus ponens
If S then P
And S
So P

Modus ponens
If you eat me then you'll grow up
And you eat me
So you'll grow up

2.1.2 Second model: Contradiction and dilemmas

Supporting text by Lewis Carroll (1865), *Alice's Adventures in Wonderland*, illustrated by J. Tenniel[34].

1. Analytical interpretation

The following text (Carroll, 1865, chapter 8) expresses a 'syllogism battle' between the King and the Executioner to decide, whether one can decapitate the Cheshire cat, whose head appears without the body.

[34] Carroll, L., 1865. *Alice's Adventures in Wonderland*. London: Macmillan and Co. Reprint 2006. *The complete illustrated Lewis Carroll.* London: Wordsworth Editions, pp. 82–83.

TABLE 22

SYLLOGISM BATTLE

> 'The executioner's argument was, that you couldn't cut off a head unless there was a body to cut it off from: that he had never had to do such a thing before, and he wasn't going to begin at his time of life.
>
> The King's argument was, that anything that had a head could be beheaded, and that you weren't to talk nonsense.
>
> The Queen's argument was, that if something wasn't done about it in less than no time she'd have everybody executed, all round. (It was this last remark that had made the whole party look so grave and anxious.)'
>
> Lewis Carroll, ed. 2006, p. 83

The Executioner's argument is a hypothetical syllogism in the form of the *modus ponens:* if p, then q, and p, then q. According to this way of thinking, the executioner's conclusion is valid.

The King's argument is a categorical syllogism called an enthymeme, because it lacks a step in the reasoning. One of the premises is evaded and held for certain. If we introduce the implicit premise, the King's syllogism is of the classical form (named Barbara), All M is P and S is M, thus S is P (with S = cat, P = beheaded and M, the middle term = head). The King's implicit conclusion seems valid. This leads to a paradox. The reasoning is correct, but the conclusions of the King and the Executioner are contradictory. Either it is true the cat can be beheaded, or it is true the cat cannot. Whatsoever, the two conclusions cannot be true at the same time.

Rather than trying to solve this paradox, the Queen avoids it. She prefers to show authority. However, the argument of authority belongs to rhetoric and not to logic (dialectic). It does not solve the paradox. From Alice's point of view, this is a dilemma. If she agrees with the Executioner, she gets angry with the King. If she agrees with the King, she gets angry with the Executioner. Worse, in the absence of being able to solve the paradox, the Queen, as usual, will shout in both cases, 'Off with his head!'.

The dilemma is part of 'the five indemonstrable syllogisms of Chrysippus'. It is a disjunctive compound syllogism. It introduces several logical connectors such as the exclusive 'or', either one or the other, if … then. Lewis Carroll's text can be interpreted as what logicians call a 'complex constructive dilemma'.

TABLE 23

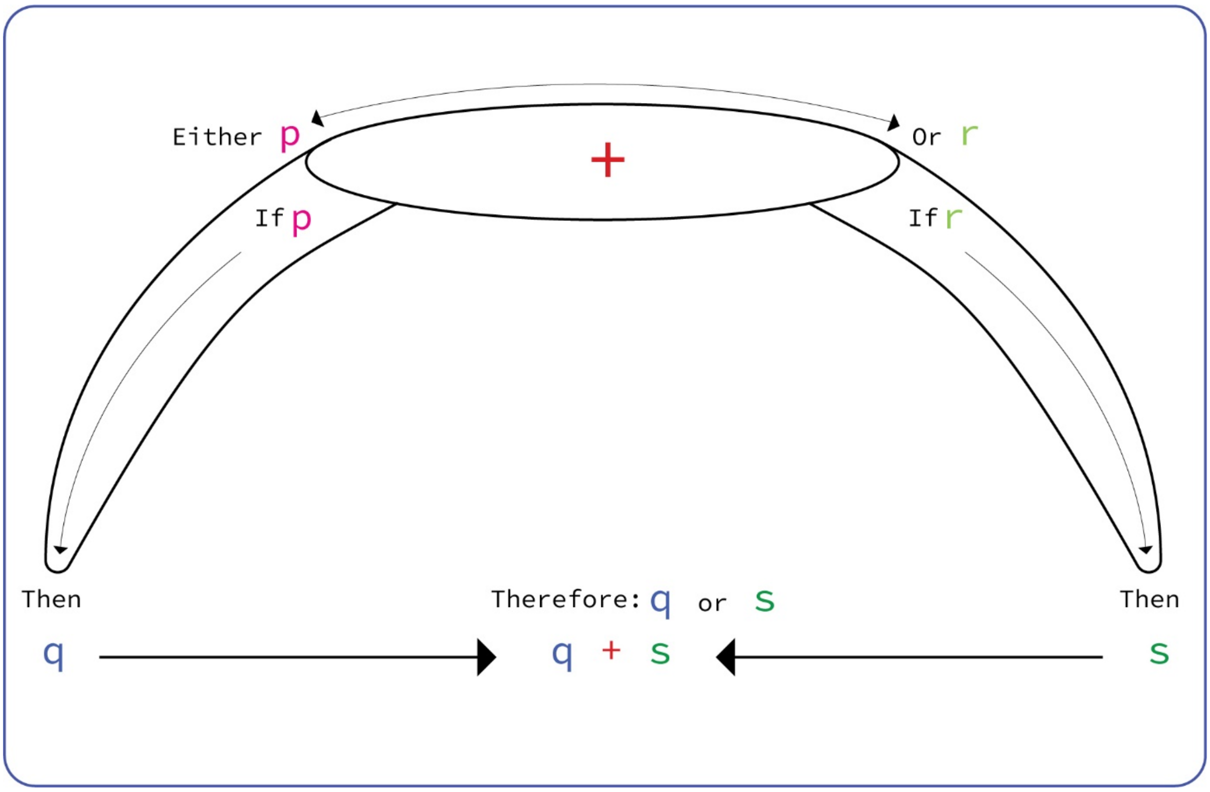
A COMPLEX CONSTRUCTIVE DILEMMA

Either Alice agrees with the King (noted here p), or she agrees with the Executioner (r).

If she agrees with the King (p), then the Executioner will be angry (q).
If she agrees with the Executioner (r), then the King will be angry (s).

Conclusion, whatever the decision is, either the Executioner (q) or the King (s) will be angry with her.

This compound syllogism is only implicitly expressed in the tale. Confused by the dilemma; to avoid ending up as the Buridan's donkey, who died of hunger and thirst because he couldn't choose between a peckish oat bran and a bucket of water, Alice will only divert the conversation. This cat, 'It belongs to the Duchess; you'd better ask *her* about it,' she simply says. As proposed by Kreeft (2004, reprinted 2014, p. 306), this dilemma can be illustrated by the two 'horns' to which are added here the symbolic letters p, q, r, s and the connectors or, noted '+', and If… Then:



Another way of approaching the dilemma is to transcribe it into a logical equation. One might then ask whether the 'Either p or r 'of the alternative is exclusive or whether there are other alternatives, in which case the dilemma would disappear.

TABLE 24

THE DILEMMA IN EQUATION FORM

With the following symbols:

p = Alice agrees with the King.
r = Alice agrees with the Executioner.
q = The executioner will be angry.
s = The King will be angry.

With 'inclusive or', denoted +; 'implies' denoted =>, 'and' denoted '.' and the 'conclusion' noted ∴,
the formula of the complex constructive dilemma will be written in the form of a logical equation:

$[(p + r) . ((\text{if } p => q) . (\text{if } r => s))] \therefore q + s$

The question then is how to illustrate a paradox and a dilemma.

2. Preliminary pop-up: The Cheshire Cat Paradox

What characterises paradoxes and dilemmas is the introduction of premises that are apparently reasonable, but which lead to a contradiction that generates absurd consequences, called paradoxes, or from which there is no escape, called dilemmas. Who is right, the King or the Executioner? To make children aware of the principle of contradiction, I constructed a pop-up: The Cheshire Cat Paradox (Preliminary Game 2), where they can give their answer and see the effect produced.

2.1.3 Third model: Solving a two-equation system

Supporting text by Lewis Carroll (1885). *A Tangled Tale. Knot 7, Petty Cash*[35]*.*

1. Understanding of mathematical text.

Contrary to previous logical texts, Lewis Carroll here explains clearly how to solve a two-equation system. Thus, it is not a question here of interpreting a text, but of understanding it. Narrated in the form of a short story, he proposes to solve the following system of equations using algebra.

[35] Carroll, L., 1885. *A Tangled Tale.* London: Macmillan and Co. Reprint 2006. *The complete illustrated Lewis Carroll.* London: Wordsworth Editions, the story, pp. 987–990. Answers Knot 7, pp. 1024–1035.

TABLE 25

THE LEWIS CARROLL KNOT 7

Knowing that the price of 1 glass of lemonade, 3 sandwiches and 7 biscuits cost 1s. 2d. and that 1 glass of lemonade, 4 sandwiches and 10 biscuits cost 1s. 5 d., Lewis Carroll asks children to find the cost of:

1 glass of lemonade + 1 sandwich + 1 biscuit
2 glasses of lemonade + 3 sandwiches + 5 biscuits

If x denotes the price – in pence – of a glass of lemonade, y that of a sandwich and z that of a biscuit, the problem put in the symbolic form of a system of two equations is written:

$x + 3y + 7z = 14$ and
$x + 4y + 10z = 17$
(1 shilling is 12 pence)

The question posed is
to find out what the price of:

$x + y + z = ?$
$2x + 3y + 5z = ?$

The letters x, y, z are a useful invention ascribed to François Viète (1540–1603) and René Descartes (1596–1650). The concept is astonishing. It assumes the unknown x, y, z to determine the known. Lewis Carroll makes extensive use in his diagrams of the letters x, y, m and their complementary terms not x, not y, not m, respectively noted x', y', m'. As taught at school, he uses the letters x, y, z to solve mathematical equations. However, he proposes here to children to solve a system of two equations with three unknown x, y, z.
The Carrollian trap.
The children's first instinct will certainly be to try determining the unit price of a glass of lemonade, the price of a sandwich and the price of a cookie. Of course, they will not find a solution. Lewis Carroll has set a trap for them. They will learn by making mistakes that they cannot solve a system of two equations in which there are more unknowns (x, y, z) than equations (2 here). The idea of learning through error correction is a modern idea that can be found in the field of artificial intelligence.

TABLE 26

THE LEWIS CARROLL KNOT 7. SOLUTION

What can be calculated is the value of certain combinations such as $x + y + z$ and $2x + 3y + 5z$, not x, y and z taken individually, on the condition that one of the three unknowns can be eliminated during the demonstration.

As usual, Carroll provides the solution in the appendix and comments on the mistakes made by the children who completed the exercise.

The answer to Knot 7 is 8 d and 1 s. 7d:

1°) $x + y + z = 8$ d
2°) $2x + 3y + 5z = 19$ d, or 1 s. 7d

The question that remains to be answered is: How to illustrate these algebraic equations and find the solution by means of the visual arts?

2. Illustration of algebraic equations.

The first idea that comes to mind is to draw 2 glasses of lemonade, 3 sandwiches and 5 biscuits, then combine them with a plus sign (+) and put a question mark (?) after the equal sign (=). This is what is done nowadays in a comic such as a BD to learn arithmetic and algebra (Gonick, 2015). However, this type of illustration has two disadvantages that can be pedagogically discussed.

The first drawback is that it does not allow visualising the reasoning that leads to the solution of the problem. The letters x, y and z, will have only been translated into images. Because pure mathematics and logic are abstract and formal, the solution of the problem has nothing to do with the image of a glass of lemonade, sandwiches and biscuits. What is essential to see and make understood is that the solution of the two equations (here: $x + y + z$ and $2x + 3y + 5z$) will remain the same if we replace the glass of lemonade with a glass of orange juice, sandwiches with pastries and biscuits with yogurt pots. The important concept here is that the solution is not related to the nature of the objects (x, y and z). The solution depends exclusively on how its objects are linked together by the symbols of the connectors (addition, multiplication, equality) and the conjunction of the two equations. In other words, the answer to the problem posed depends on the axioms and the rules of arithmetic and algebra.

The second drawback is that drawing a glass of lemonade, sandwiches and biscuits may divert attention from the intended purpose. The mathematician Oliver Byrne[36] (1847) realised this and in his illustration of the *Elements of Euclid*, he chose not to represent objects, but to symbolise them with shapes and colours (squares, triangles, half-moons, etc.). These symbols are easily identifiable and above all in accordance with symbolic and abstract logic.

As suggested by the pedagogical method created by Maria Montessori (1870–1952) at the beginning of the 20th century and used nowadays in many countries[37], it is possible to encourage children to employ wooden figures of different colours (round, square, rectangle) to move them on a game board. According to this method, moving geometrical figures helps to develop mathematical reasoning. By associating this method with the one of Byrne, it gave me the idea of using red circles, yellow rectangles and blue squares to solve the problem, following step by step the solution proposed by Lewis Carroll ('Answer to Knot 7'). From this perspective, the two equations could be put in the following form.

[36] Byrne, O., 1847. *The First Six Books of the Elements of Euclid*. London: William Pickering Editions. Reprint 2014 and 2017. Köln: Werner Oechslin and Taschen GmbH Editions.

[37] Esclaibes S.d., and N. d'Esclaibes, 2019. *100 activités Montessori, 6-12 ans*. Paris: Hatier. Introduction, pp. 5–10.

TABLE 27

VISUAL RESOLUTION OF A SYSTEM OF EQUATIONS

The resolution of the system of equations is done by using coloured circles, squares and rectangles instead of the letters x, y, z.
Resolving:

Let's eliminate circles ● and rectangles ▬

and reduce everything to several squares ■

1° Subtract the first equation from the second:

1●	+	4▬	+	10■	=	17
-1●	-	3▬	-	7■	=	-14
		1▬	+	3■	=	3

Which is equivalent to:

1▬ = 3 - 3■

2° Replace the rectangle ▬ with its equivalent
3 - 3■ in the first equation:

1● + 3▬ + 7■ = 14

is equivalent to:

1● + 3 x (3-3■) + 7■ = 14

is equivalent to:

1● + 9-9■ + 7■ = 14

that is:

1● = 5 + 2■

3° Replace the circle 1● and the rectangle 1▬
by their equivalent value in the requests. Hence the solutions:

1° solution:

1● + 1▬ + 1■ = (5 + 2■) + (3-3■) + 1■= 5 + 3 + (2■ + 1■ - 3■) = 8

Because the sum of the squares ■ is zero.

2° solution:

2● + 3▬ + 5■ = 2 x (5 + 2■) + 3 x (3-3■) + 5■ = 10 + 4■ + 9-9■ + 5■
= 10 + 9 + (4■ + 5■ - 9■) = 19 Because 4■ + 5■ - 9■ = 0

Hence the answers to Knot 7:
1 glass of lemonade + 1 sandwich + 1 biscuit = 8 d
2 glasses of lemonade + 3 sandwiches + 5 biscuits= 19 d, or 1 s. 7d.

In practice, I apply this principle of geometrically shaped coloured counters for the resolution of Lewis Carroll diagrams (Game 6), then for the resolution of compound syllogisms using Truth Tables and Boolean algebra (Game 7). It should be noted that the syllogisms to which Lewis Carroll explicitly refers in *The Game of logic* and *Symbolic logic* are, such as in Venn diagrams (Game 5), 'categorical syllogisms' originally highlighted by Aristotle and illustrated here in Puzzle games 1 to 3.

To conclude on this point.
This approach encourages distinguishing between two languages: on the one hand, the language of deductive reasoning allows for a discourse to progress from arguments to a conclusion, on the other hand, axioms and rules which serve to construct reasoning and to judge its validity. This language composed of axioms and rules, which can be designated a 'metalanguage', must also be highlighted and illustrated because these two languages do not go one without the other. Moreover, if the aim is to use the visual arts to solve syllogisms and algebraic equations, this requires building a bridge between the visual arts and the pure sciences (algebra, geometry and logic) that have their own language and metalanguage. To resolve this question, my research has conducted me to take an interest in Leonardo da Vinci's conceptual method.

2.2 Leonardo da Vinci's conceptual method: from 'Practice-Based Research' to 'Practice-Led-Research'

Leonardo da Vinci's approach explores two concepts.
The first is practice-oriented. It integrates the reasoning into the drawing from numerous experiments. This involves the construction of models, prototypes, adapted tools, maps and puzzles. This approach is similar to that of 'Practice-Based Research'. He makes a contribution to the development of knowledge and truths through the creative practice of drawing.
Secondly, which is more theoretical he introduces concepts and rules of geometry, such as those of perspective or the golden ratio, into his drawings. This way of thinking corresponds more to the conceptual idea of 'Practice-Led-Research'. I will deepen this approach in the second part of my research.
Bringing together artistic disciplines, scientific reasoning, theory and practice was Leonardo da Vinci's (1452–1519) idea, but it was also that of painters, scientists and architects who around him. From this point of view, his passion for mathematics and Euclidean geometry brings him closer to Lewis Carroll.

He exposes his method in his various treatises, notably in his treatise on painting *Trattato della pittura*, or *Codex Urbinas*[38]. His method can illustrate complex reasoning and concepts. It seeks to understand what it means to 'demonstrate', prove the truth or 'refute', as mathematicians do. This does not prevent him from having faith in the artist's experience, freedom and creativity. His experiences serve to validate his theories. His concept of 'refutability' announces through art what will be the foundations of the theory of the philosopher of science, Karl Popper (1902–1994). For Popper, a theory is only scientific if it can be disproved (Popper, 1963).

2.2.1 Leonardo da Vinci's method

His method consists of three steps (Brioist, 2019). First, he experiments. Then he conceptualises and theorises. Finally, he tests the theory.

[38] This treatise is kept in the Vatican Library: Brioist, P., 2019. *Les audaces de Léonard de Vinci*. Paris: Stock Éditions, p. 102, note 19.

TABLE 28

SUMMARY OF THE THREE STAGES OF LEONARDO DA VINCI'S METHOD

First Step. Practice and Creativity

At the base of the practice are creativity and experimentation. He summed up in two word his method of drawing: componimento inculto, the 'uncultivated composition' (Brioist, pp. 401–402). According to him, we must first trust: intuition, creativity, imagination, analogical thinking, earthly and sublunary similarities, waking dreams and fables, learning by mistake. He is convinced that in painting the perfect form is unconsciously present in mind, likely to emerge instinctively in a harmonious way. More wisely, he is convinced that the practice of drawing can itself be a source of scientific discoveries. His practice is that of the inventor, by the construction of military machines, puppets, automatons, models and prototypes. This is how he created a helicopter prototype, and in particular a robot[39]. This gave me the idea of building prototypes, nine in total.

Step 2. Theory

After this first step, the practice must be based on a solid theory. For him, the two enemies to confront are, scholasticism (purely abstract reasoning, without confrontation of experience and tests), and practice without theory[40] Leonardo da Vinci's theory consists in applying the laws of perspective, those of the golden ratio of the Divine proportion, or those of the combination of colours and shapes, studying, for example, all types of noses, all types of movements. He owes these theories to the scholars and artists he frequents or whose works he reads: Vitruvian, Archimedes, Euclid, Aristotle, Francisco di Giorgio, Brunelleschi, Alberti, Piero della Francesca, Luca Pacioli[41]. He will even illustrate the treatise of the mathematician Pacioli on the Divina proportione (1509). According to Leonard da Vinci the theory of the proportions of the human bodies also applies to the construction of monuments, with an ideal of beauty, usefulness and efficiency. He attaches importance to the theories of Marcus Vitruvius Pollio, known as Vitruvius, a Roman architect who lived in the 1st century BC, author of a treatise on architecture, entitled De architectura. For Vitruvius, temples must be pleasant to the gods and, for buildings to remain stable, mathematical calculations are necessary. The triad of three criteria to be respected is firmitas (solidity), utilitas (utility), venustas (beauty)[42]. Hence the idea of introducing these criteria in the validation of an image/text ratio.

Third and Last Step. Testing

The conclusion is the testing of the theory. Although his small-scale models do not always work in real scale, as scientists nowadays do, they are useful for testing theories. This requires the prior definition of criteria for judging the validity of a theory and its practical results. My research will lead me to select nine categories of criteria and to consider how they can be combined (Chapter III and part II).

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[39] Leonardo's robot or Leonardo's mechanical knight, around the year 1495, was a humanoid automaton. He is wearing medieval armour. He could perform various movements: sitting down, waving his arms, etc. For his robot, he uses a system of pulleys, gears and cables (the motor does not yet exist).

[40] (Brioist, 2019, pp. 258, 266–267, 272–273 and notes 87 and 120.)

[41] The fresco of the Holy Trinity by the Florentine painter Masaccio (1401–1428) was the first to have respected the principles of geometric perspective. It was theorised by the Florentine architect and sculptor Filippo Brunelleschi (1377–1446) and the architect Léon Battista Alberti (1406–1472). Alberti's theoretical treatise *De Pictura* marked the beginning of the pictorial Renaissance (Brioist, 2019) and (Andersen, 2007).

[42] Brioist, 2019, pp. 111–123, and pp. 235–236.

Since Leonardo da Vinci's method is proven, the idea is to draw inspiration from two angles. Firstly, this method has a pedagogical interest. As Brioist (2019, p. 419) points out, Leonardo da Vinci was an excellent teacher. Secondly, it is not enough to have a method (or theory), it is also necessary to be able to transmit it. This is an example of a criterion that can be taken into account. A method is really useful if it can be shared. Hence the need of writing and illustrating detailed instruction manuals to accompany each game. The point to be retained here is that Leonardo da Vinci succeeded in introducing reasoning into art, in particular through the theory of perspective and the golden ratio. The following paragraph examines the relationship between art and geometry, i.e. the relationship between geometrical figures (or images) and texts that express reasoning.

2.2.2 Three examples of Visual Creative Thinking Art: The Puzzle of Pythagoras' theorem, the Golden ratio and the Fibonacci puzzle

The language of geometry is probably the closest language to drawing. It establishes a link between drawing and a scientific text. It is used to express through text and images the theory of demonstrations and proof. In the pure tradition of the writings of the *Elements of Euclid* (circa 300 BC), it is composed first of axioms, definitions and rules (metalanguage) to which is associated a symbolic language composed of signs and mathematical operators. Illustrating a demonstration, using geometric figures or algebraic signs, is similar in a way, to illustrate an instruction manual. It is a question of illustrating reasoning, which makes it possible to establish a theorem and to prove its validity. In the field of logic, especially in syllogism theory, logicians talk rather about determining valid conclusions than establishing theorems, but the idea is the same. This gave me the idea of writing Booklets and illustrating them to explain the rules of the games and show step by step how logic works. In the next two examples, the first, the Pythagorean theorem shows in a visual way how to calculate the length of the hypotenuse of a right-angled triangle. The second example, the golden ratio, shows how to establish an ideal visual proportion. According to the adepts of the golden ratio, the aim is to define an ideal aesthetic criterion, whether it is to create a work of art, build a pyramid or define the dimensions of a book cover.

1. The Puzzle of Pythagoras' theorem

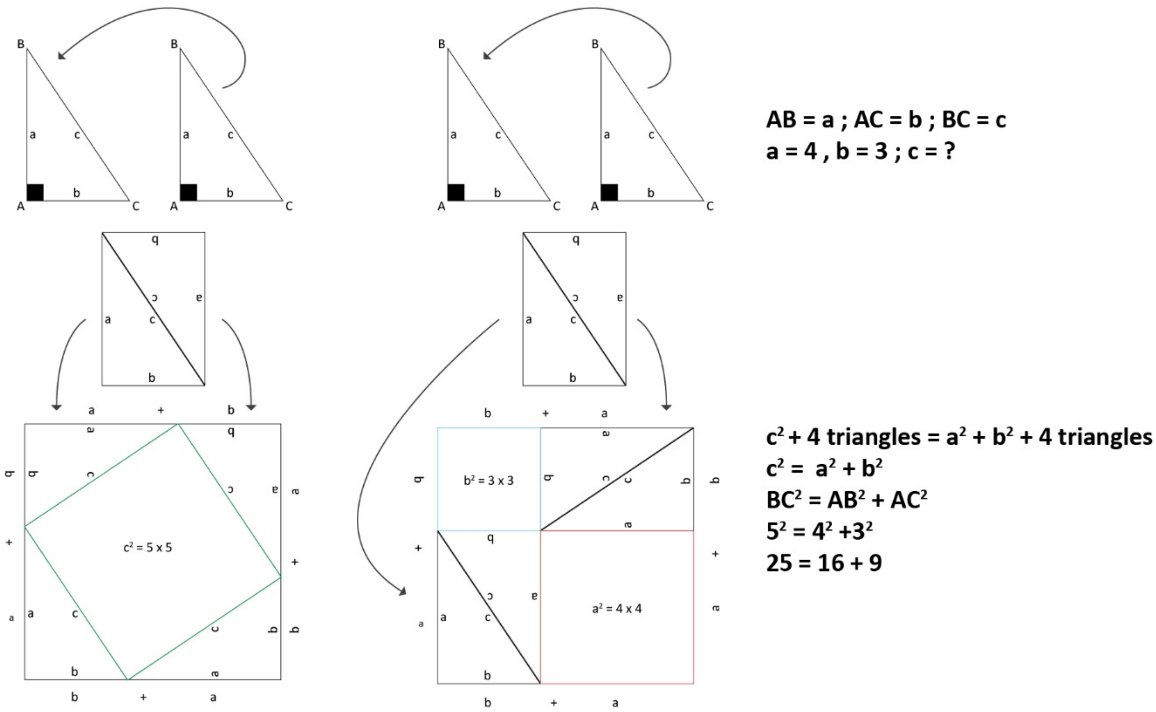
The demonstration and illustration is done in two steps.
First step: The demonstration of the theorem.

In the following drawing, the triangles and squares illustrate the notorious proposition 47 of the *Euclid Elements, book I*, even if it is not the rigorous and original demonstration of Pythagoras' theorem[43].

TABLE 29

THE PYTHAGOREAN THEOREM DEMONSTRATED BY THE DRAWING

Draw a right triangle at A, side AB = 4 cm, and AC = 3 cm. The problem is to find the length of the BC hypotenuse by simply moving and pivoting the geometric figures.



According to Pythagoras' theorem, in a right-angled ABC triangle in A (of hypotenuse BC), the square on the side opposite to the right angle (BC^2) is equal to the sum of the squares of the other two sides ($AB^2 + AC^2$). Hence the equality $BC^2 = AB^2 + AC^2$.

Application: if AB = 4 cm and AC = 3 cm, then $BC^2 = (4 \times 4) + (3 \times 3) = 25$ and BC = 5 cm.
This theorem has many applications, in analytical geometry (measurement of distances in a 4-dimensional tesseract), in geometry in space (calculation of distances), etc.

[43] For a demonstration of the theorem, see among other authors, Davis, P.-J., and R. Hersh, 1982. *The mathematical experience*. Boston: Birkhäusser. Translated into French by Bordas, 1985. *L'Univers mathématique*. Paris: Bordas, pp. 140–144, and the original visual presentation in Euclid, 300 BC. Reprint 1956. *The thirteen books of The Element*. Translated with introduction and commentary by Sir Thomas L. Heath, vol. 1, books I and II, 2nd ed. New York: Dover Publications, Inc., pp. 349–350, and illustrated by Byrne, 1847. *The first six books of the Elements of Euclid*. Reprint 2017. Köln: Werner Oechslin and Taschen GmbH, pp. 48–49.

Despite the existence of non-Euclidean geometry (not examined here: Lobachevski, 1829, Riemann, 1867), Lewis Carroll remains attached to the language of Euclidean geometry. Presumably, to follow the evolution of his time, he sought, through syllogisms, to bring algebra and geometry closer together. This is what Descartes had done by inventing analytical geometry from its Cartesian coordinate system[44]. The idea of the xx' and yy' axes will be found in Lewis Carroll's diagrams to solve syllogisms.

Second step: illustration of Pythagoras' theorem.
The Puzzle of Pythagoras' theorem reconstitutes here the demonstration of the theorem from the assembly of the wooden pieces of a jigsaw puzzle.

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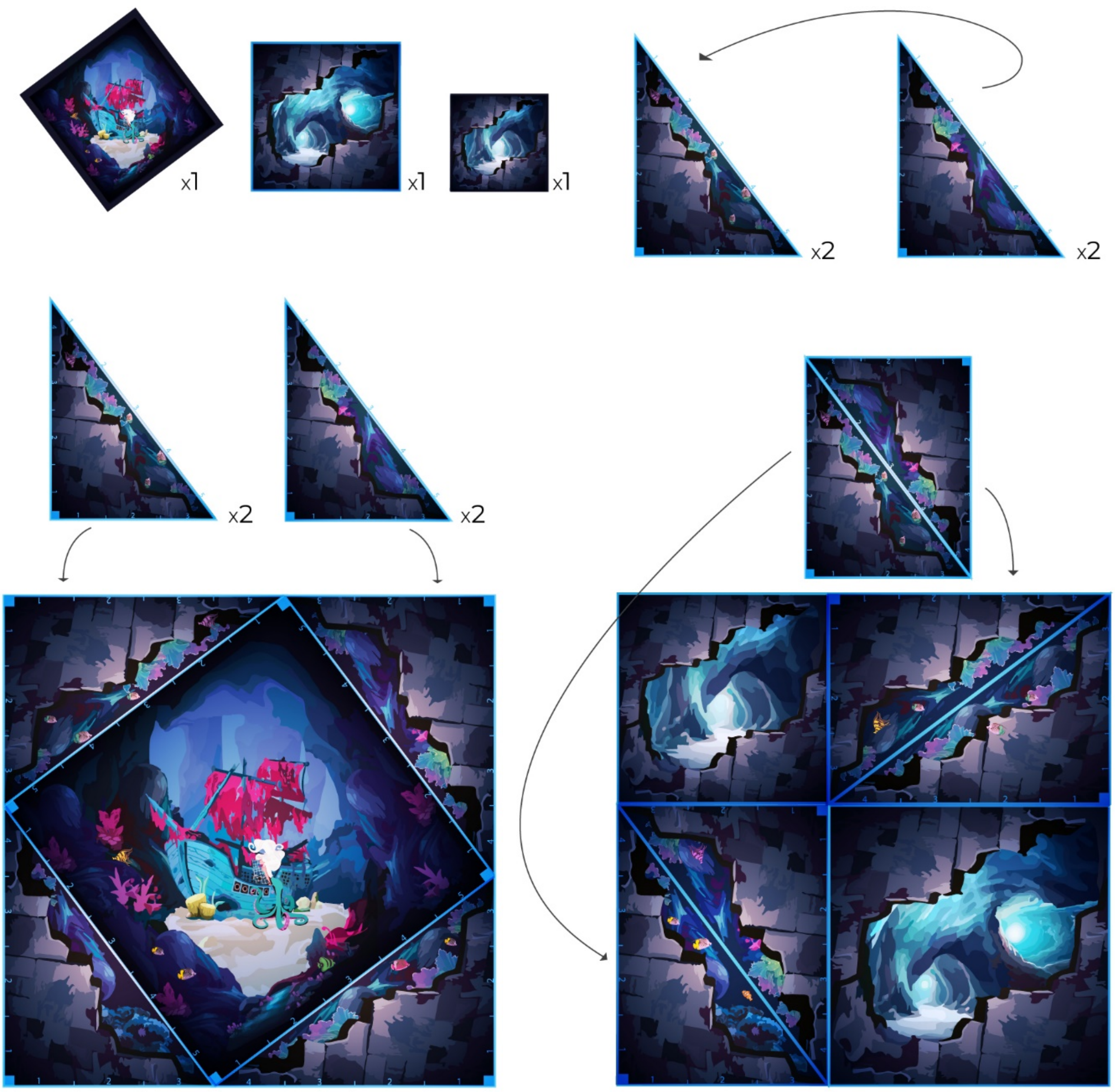
[44] The Cartesian coordinate system is composed of two axes xx' and yy' which intersect at the centre O, making it possible to locate two points A and B by their coordinates: x and y are positive numbers (1, 2, 3, etc.); x' and y' are negative numbers (-1, -2, -3 etc.). The Pythagorean theorem is used, for example, to calculate the distance between points A and B.

TABLE 30
PERSONAL ILLUSTRATIONS

PYTHAGORAS' THEOREM PUZZLE

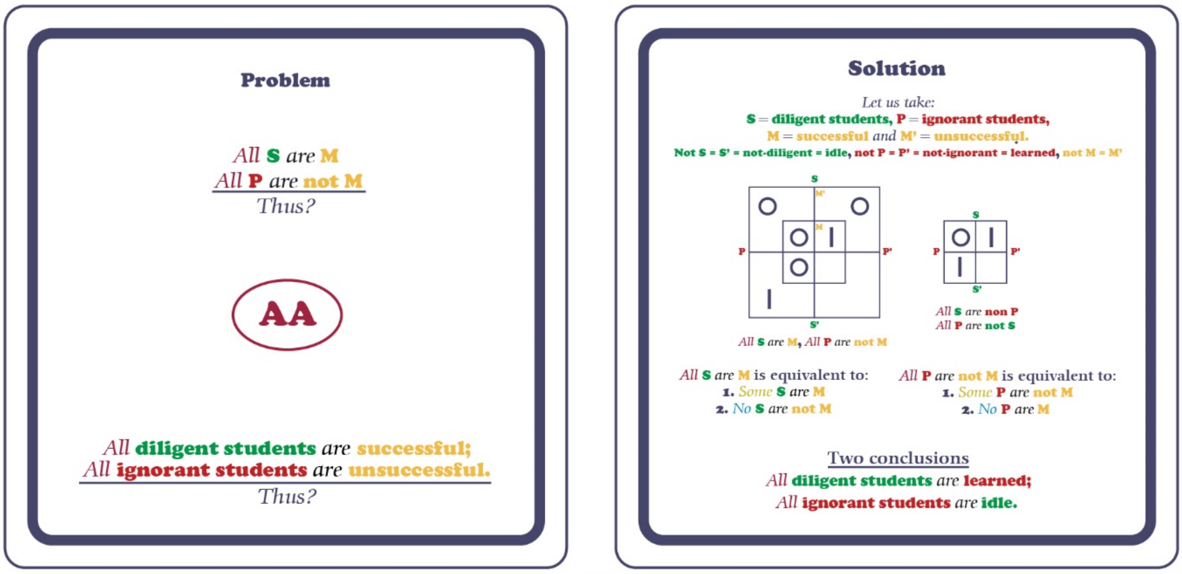
Elements' given: 3 illustrated squares, 8 illustrated rectangular triangles



The moving of the rectangle triangles within a square makes it possible to find the formula of Pythagoras' theorem[45]. This is an example of the link between geometry, algebra and arithmetic. The tactile assembly of wooden triangles allowing visualisation of the construction of the Pythagorean theorem gave me the idea to construct puzzles in Games 1 to 3 to solve Aristotelian syllogisms. In these games, children can find the conclusion of a categorical syllogism by manipulating wooden jigsaw puzzle pieces. It is an opportunity to assimilate through sight and touch three fundamental geometric concepts used in Game 5: translation, symmetry and rotation. In Game 6, based upon Lewis Carroll's diagram, I use the idea of coloured counters, Cartesian axis xx' and yy' to visually solve the categorical Carrollian syllogisms which originated with Aristotle's idea. For example, in *Symbolic logic,* when Lewis Carroll asks what would be the conclusion of the following syllogism[46], I present the problem in the form of a playing card.

PERSONAL ILLUSTRATION

CATEGORICAL CARROLLIAN SYLLOGISMS

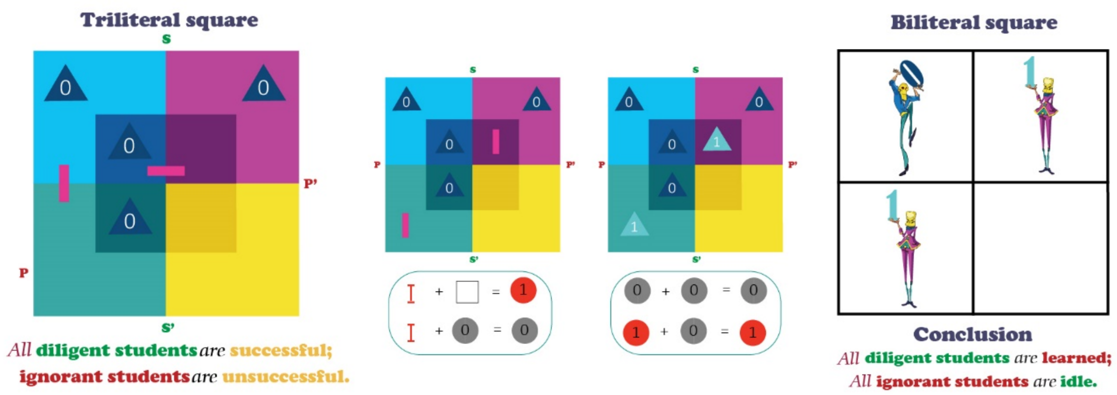


[45] Euclid. Proposition 47: 'In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.'

[46] Carroll, L., 1896. *Symbolic Logic. Part I: Elementary*. London: Macmillan and Co. Reprint 2015. *Mathematical Recreations of Lewis Carroll. Symbolic Logic. Game of Logic. Two Books Bound as One*. New York and Berkeley Enterprises: Dover Publications, Inc., pp. 62–63. Note: To establish a link with the puzzles in Games 1 to 3, the original letters x, y, x', y', m, m' are replaced here by S, P, not-S, not-P, M, not-M respectively.

To find the solution, the axes xx’ and yy’ make it possible to separate four squares. As explained in Booklet 6, x can designate in a sentence the subject (S), y the predicate (P), x’ the negation of the subject (not S), y’ the negation of the predicate (not-P). The solution written on the back of the playing card is obtained by using a simple but special addition table of 0 and 1. In game 6, the conclusion is obtained with the use of counters and figurines placed on the game board of a pop-up game.

TRILITERAL AND BILITERAL DIAGRAMS



This leads to two conclusions: All x are y’ and all y are x’, recoded in a more traditional way: all S are P’ and all P are S’, which transcribed into everyday language, are:

‘*All* diligent students are (not-ignorant, i.e.) learned;
All ignorant students are (not-diligent, i.e.) idle.’

2. The Golden ratio

Since ancient times, geometry has established a substantial relationship between art and science. During the Italian Renaissance (1300–1600), painters, including Leonardo da Vinci, rediscovered literature, philosophy and the sciences of antiquity, and in particular the golden ratio in art. Associated with the sequence of numbers from the Italian mathematician Leonardo Fibonacci (1170–1250), the golden ratio is an example of an ‘ideal’ ratio. This makes it an aesthetic criterion. It deserves attention for at least five reasons set out in the following table.

TABLE 31

FIVE REASONS TO BE INTERESTED IN THE GOLDEN RATIO
AND THE FIBONACCI SEQUENCE OF NUMBERS

1° The Fibonacci sequence is the example of a mathematical syllogism whose conclusion can be presented in the form of an image. The sequence converges towards the golden ratio.

2° As for vanishing points in perspective, it can be used as a tool for drawing.

3° Children can make from it an educational puzzle, both for drawing and reasoning.

4° The golden ratio has become an aesthetic criterion that links science and Art.

5° It raises in Art more debates on Beauty and its enigmas[47].

Called Phi (ϕ) from the 21st letter of the Greek alphabet, in homage to the art of the sculptor Phidias, the golden ratio has become as known as Pi (π) = 3.14159 … or square root of two (√2) = 1.41421… It can be calculated and drawn. In Geometry, such as in Art, its value corresponds to the ratio between two lengths: a (the largest) and b (the smallest), such that (a + b)/a = a/b. There are several ways to establish a relationship between two quantities as the arithmetic mean (a + b)/2, e.g., the average of 7 and 9 is (7 + 9)/2 = 8. The geometric mean is a/c = c/b or c = √ab. Since ancient times, however, it has been considered that the most harmonious way of dividing two segments of lines into the 'extreme and mean ratio' is the one proposed by Euclid (Book IV, proposal 10): (a + b)/a = a/b.

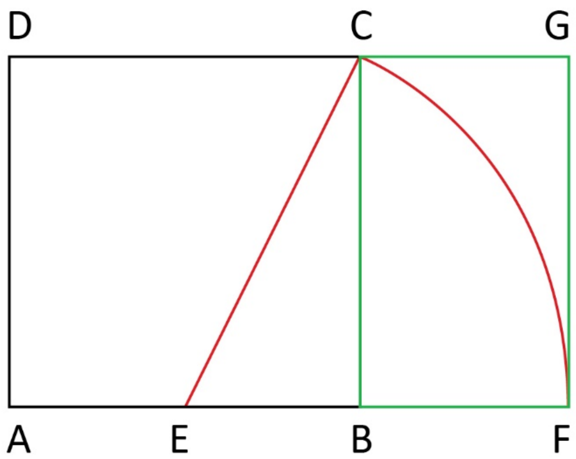
One way to teach children how to build this "ideal" ratio is to show them that they can construct a golden rectangle in a few steps, using a ruler and a compass. This construction will show them how to associate drawing with geometric reasoning.

[47] Darriulat, J., June 2019. La beauté et ses énigmes. Paris: *Papiers, revue de France Culture*, dossier 29, pp. 31-38.

TABLE 32

CONSTRUCTION OF A GOLDEN RECTANGLE

1° Draw a square ABCD
2° Take the middle E on the AB side
3° Draw the EC segment from E to the opposite vertex C
4° Take the distance EC as the distance between the compass
5° Draw from point E a circular arc of radius R = EC
which intersects the extension of the line AB at point F.
6° Draw the line FG, perpendicular to AF and parallel to BC which cuts the extension of the DC line into G.
7° Point F is the third vertex of the AFGD golden rectangle. This makes it possible to construct the following figure.



This construction makes it possible to understand the difference in reasoning between the pure sciences and the experimental sciences. Suppose that while drawing, one constructs this figure by chance. Who guarantees us this construction is indeed that of a golden rectangle? It is necessary to provide irrefutable and universal proof of this by a mathematical demonstration.

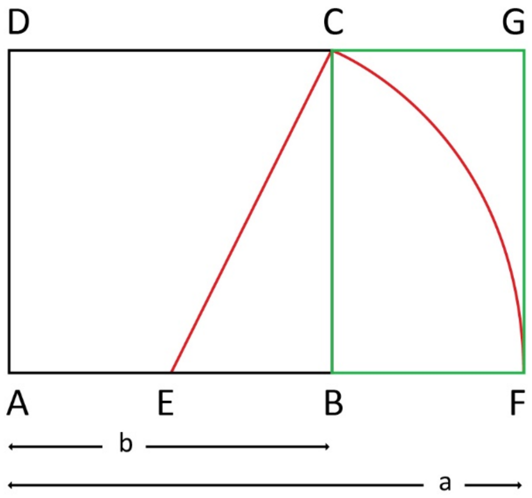
TABLE 33

GIVES AN EXAMPLE OF THE ART OF THE DEMONSTRATION

1. By definition, the golden ratio, noted (ϕ) is equal to $(1+\sqrt{5})/2$ or 1.618… A rectangle is a golden rectangle if the ratio of the measurements of its length (L) and its width (l) is equal to the golden ratio, i.e. $L/l = \phi = 1.618\ldots$
2. The problem: it must be proven that the AF/AB ratio $= \phi$.
The demonstration is easily done using the Pythagorean theorem.
3. Demonstration. To simplify the entries, let us designate ‘a’ as the length of the AFGD rectangle such that:
$a = AF$ and $b = AD = AB = BC$

1° In the square ADCB: $AD = AB = BC = b$ and that the four angles are right (90°), and that by construction.
2° $EB = EA = ½ AB = (½) b$,
3° radii of the same circle of centre E are equal: $EF = EC$,
4° $AF = a = AE + EF$.
According to Pythagoras’ theorem, in the rectangular triangle EBC:
$(EC)^2 = (EB)^2 + (BC)^2$, i. e. $(EC)^2 = (1/2b)^2 + b^2 = (5/4)\, b^2$.

Taking the square root on both sides of the equation, we have: $EC = (1/2\sqrt{5})\, b$.
$AF = AE + EF$. However $AE = EB = 1/2b$ and $EF = EC = (1/2\sqrt{5})\, b$.
So $AF = 1/2b + (1/2\sqrt{5})\, b = [(1+\sqrt{5})/2]\, b$.



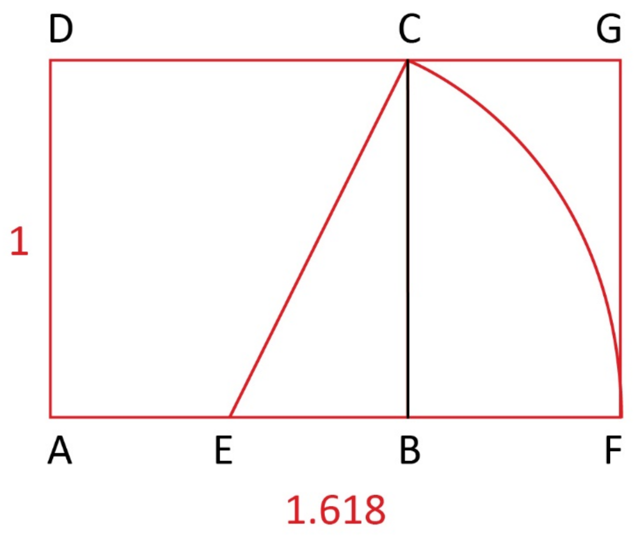
Quod erat demonstrandum: Q.E.D.,
as the mathematical adage goes.

Here is the important point. While the construction of the golden rectangle by freehand could be an intuitive or experimental discovery (with imperfectly right angles at 90°), the demonstration leads to *a certain and universal result*. It is obtained by deductive reasoning, based on mathematical rules and precise definitions. That is what mathematicians call the 'beauty' of mathematics. As Lewis Carroll explains in *Curiosa Mathematica* (Part 1, 1888, pp. ix-x)[48], the charm of mathematics lies in the absolute certainty of the results. Most other sciences are in constant change. The result of the theorem is universal.

TABLE 34

VIEWING THE GOLDEN RATIO
IF b= AD = 1 UNIT, THEN a =AF = 1.618
THIS RATIO L/l= 1.618/1 = Φ HAS BECOME AN AESTHETIC CRITERION.

Application: by taking a square ABCD on the side equal to 1, the side of the AFGD rectangle will be equal to 1.618. This makes it possible to directly visualise Euclid's proposal to cut two segments of lines into" extreme and mean ratio'.



[48] Carroll, L., 1888. *Curiosa Mathematica, Part 1*. London: Macmillan and Co., preface pp. IX-X, cited in Gattégno, 1974, p. 150 and note 2.

Some artists use the golden ratio consciously. This is the case of Leonardo da Vinci in his famous drawing of *The Vitruvian Man* with his arms spread out[49]. It is also the case of Albrecht Dürer, the mathematician painter Piero della Francesca, Michelangelo, and more recently, the architect Le Corbusier, Dali, or the mathematician and physicist Sir Roger Penrose in his tiling[50]. Others will employ it unconsciously. Different compasses exist to determine the golden point in a drawing[51]. Used to reach an objective evaluation criterion, the compass makes it possible to determine whether a drawing satisfies this aesthetic criterion. In practice, the golden ratio is found at the intersection of the horizontal and vertical lines of a rectangle, whose length and width have been divided according to the ratio 8/5 (approximately equal to the golden ratio (8/5 = 1.6). To facilitate calculations, painters divide the sides of a square into 8 equal elements and take 5 on each side, the golden point is at the intersection. To avoid a central composition, and to preserve the idea of symmetry, another rule always used in photography is the 'Golden Triangle' and the 'The rule of thirds'. Objects and characters are placed on one of the horizontal and vertical lines (the force lines) that divides the format in thirds.

3. The Fibonacci puzzle

Another way to illustrate the golden ratio is to build the Fibonacci spiral made up of integers. The demonstration is an initiatory game that dates from the 12th century. Each term, starting from 0 and 1, is the sum of the two preceding terms, and the ratio of two successive numbers (the largest divided by the smallest: 5 dived by 3, 8 divides by 5 and so on) tends, by excess or default, towards the value of the 'golden ratio': 1.666…
According to legend, Fibonacci obtained this series of numbers by taking an interest in rabbit reproduction[52]. This is an example of a deductive reasoning that can be written in the form of a mathematical syllogism.

[49] The annotated drawing *The Vitruvian Man* (34 × 26 cm) was made around 1490. It solved the enigma of writing a man both in a circle and a square. Symbol of humanism and ideal proportions, the drawing respects the ratio of the golden number: 1.618… Venice: Gallerie dell'Accademia de Venice. Cabinet of Drawings and Prints.

[50] Sir Roger Penrose, Nobel prize winner of Physic 2020, PhD in algebraic geometry from Cambridge, interested in the geometric works of the Dutch artist MC Escher, is also known for having created the 'Penrose Triangle' in 1967, an optical illusion of an 'impossible triangle' and various paving figures. Penrose 'Tilings' have an infinity of geometric variants, and some use the Golden Triangle. Penrose tiling, 28 May 2020. [online]. *Wikipedia, The Free Encyclopedia.* Available at:<https://en.wikipedia.org/w/index.php?title=Penrose_tiling&oldid=959348804> [accessed 15 June 2020].

[51] [Online] 11 June 2019. Fabrication d'un compas d'or — Bois d'Art. Available at: <http://www.boisdart.16mb.com/Bois_d_Art_fichiers/fab-compas-or.pdf> [Accessed 16 June 2019]. Not to be confused with the reduction compass.

[52] How many pairs of rabbits are obtained in one year if each pair produces a new pair every month from the third month of its existence? The Fibonacci sequence gives the answer.

TABLE 35

FIBONACCI'S SYLLOGISM

First premise.
Let's add the numbers from 0 and 1: 0; 1; $0 + 1 = 1$; $1 + 1 = 2$, $2 + 1 = 3$, $3 + 2 = 5$; $5 + 3 = 8$; $8 + 5 = 13$ and so on. This is the series of numbers: 0, 1, 2, 3, 5, 8, 13…

Second premise.
Let's divide two successive numbers, the largest by the smallest: 1/1, 2/1, etc. This makes it possible to approach the golden ratio: 5 divided by 3 are 1.666 …; 8 divided by 5 are 1.6; 13 divided by 8 are 1.625…

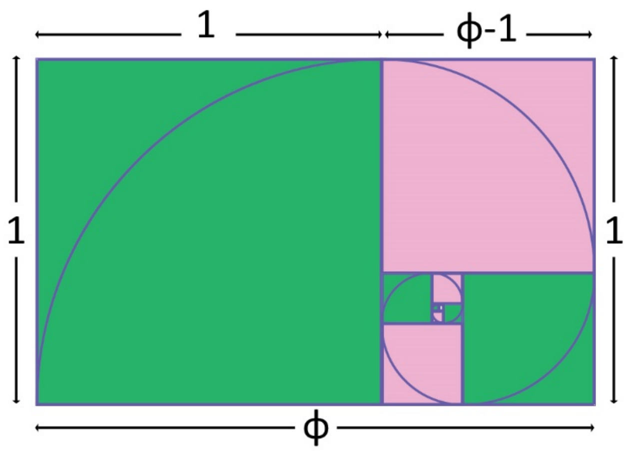
Conclusion.
From the linking of the two premises, true in themselves according to the arithmetic principles of addition and division, follows a general and universal law: the ratio of two consecutive numbers in the sequence is alternately higher and lower than the golden ratio: 1.618 033 98…

The mathematical deduction here, includes two propositions (premises) and a conclusion, which makes it a mathematical syllogism. What is interesting here is to notice the text of this syllogism can be formulated into an image, thus creating an 'ideal' image/text ratio which is that of the golden ratio. The Fibonacci spiral establishes a direct link between art and geometry: 'a mathematical beauty', writes Huntley (1970).

TABLE 36

FIBONACCI'S SPIRAL

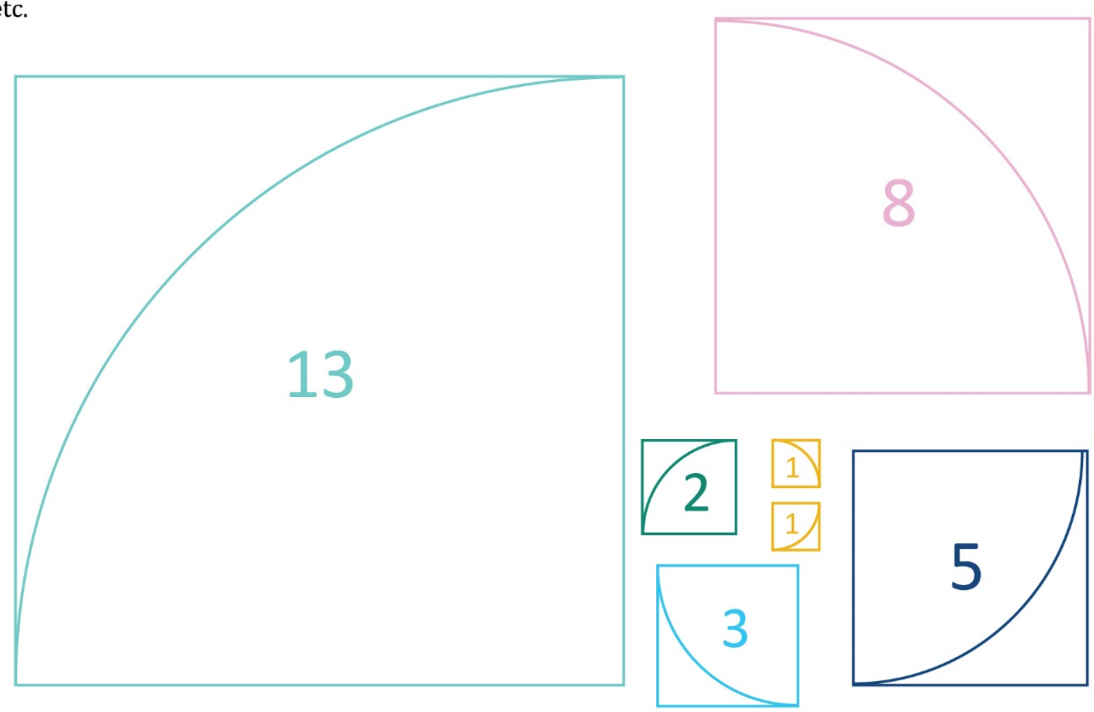
Every golden rectangle is divided into a square and a golden rectangle, which in turn can be divided into a square and a golden rectangle, and so on, to infinity. In this sequence of squares and triangles, we can inscribe a spiral that illustrates the Fibonacci Spiral.



Here I use the Fibonacci spiral as an initiatory game in the form of a puzzle and give an illustration of it. As was done in the past, this allows children to associate a drawing with geometric reasoning[53].

FIBONACCI PUZZLE

1° Draw on a cardboard a square of 1 unit on each side and an arc of a circle of equal radius 1. This is the number 1 of the Fibonacci suite.
2° Do the same to represent the second number 1 in the sequence.
3° Draw on another cardboard a square on side 2 equal to the sum of the sides of the first two squares: $2 = 1 + 1$, and draw an arc of a circle of radius 2.
4° Then draw a new side square equal to $3 = 2 + 1$ and draw an arc of a circle of radius 3.
5° And so on: $5 = 3 + 2$; $8 = 5 + 3$; $13 = 8 + 5$, etc., and draw the corresponding circular arcs: 5, 8, 13, etc.

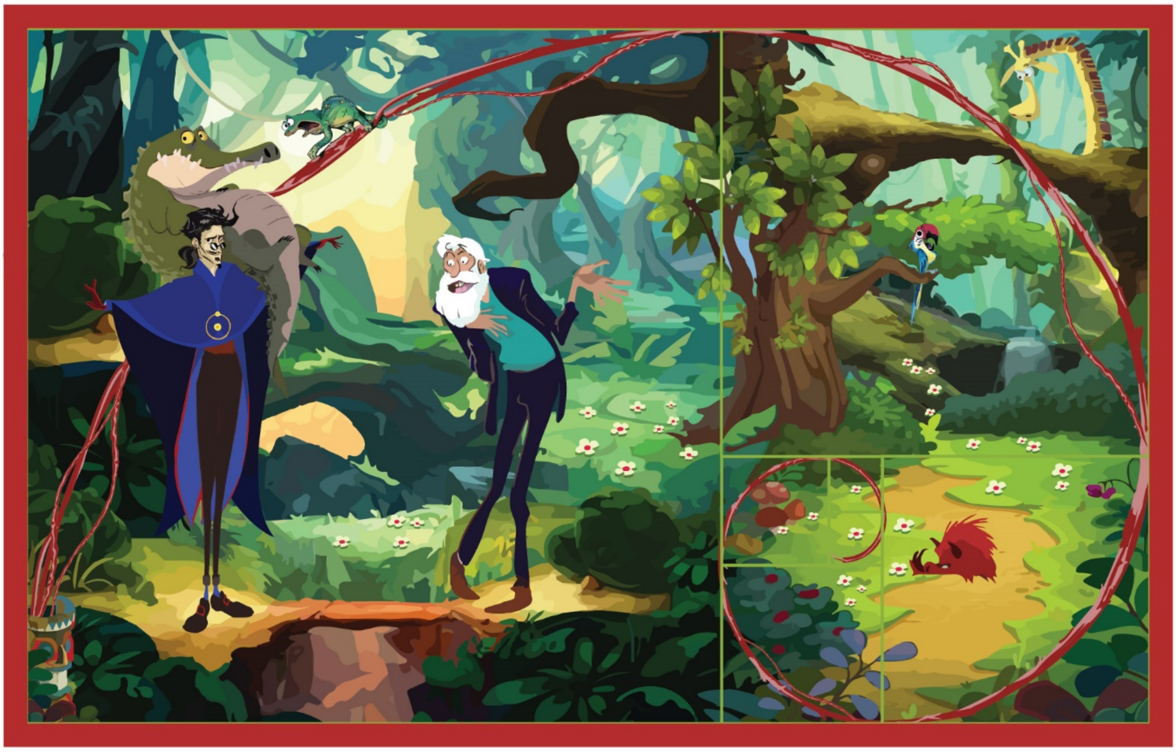


By juxtaposing the squares, children will obtain a set of golden rectangles and a visualisation of the Fibonacci sequence which will bring them closer to the golden ratio: L/l = 5/3 = 1.666…; 8/5 = 1.6; 13/8 = 1.625 …, etc. I illustrate the Fibonacci spiral with the following puzzle.

[53] They will even be able to establish a link with algebra and arithmetic. The golden ratio is the only positive solution of the second-degree equation: $\phi^2 - 1 = 0$. The solution is equal to $\phi = (1 + \sqrt{5})/2) = 1.618…$ Curiously, multiplied by itself, this number is equal to 1 plus itself: 1.618 … x 1.618… = 1 + 1.618…

PERSONAL ILLUSTRATION

THE FIBONACCI JIGSAW PUZZLES



Strangely enough, the golden ratio is an 'irrational' number. As for Pi (π) = 3, 1416 …, no formula can predict the sequence of digits after the decimal point. Like all irrational numbers, it has a mysterious and esoteric side, of a metaphysical order. For Pythagoras, these numbers had to be kept secret. For others, they are part of each of us. According to the Pythagorean tradition, 'the harmony of the Universe was a harmony of numbers[54]', the painter would have the intuition of such a ratio. This golden ratio has played a significant role in the history of art and still plays a significant role in architecture and painting. Painters seeking to imitate nature have observed that flowers, pineapples, cacti, starfish, galaxies, and other divine creations have a predilection for the Fibonacci sequence. The number of flower petals frequently corresponds to one of the numbers in the sequence. However, it should be noted this illustrates *reasoning by analogy*, A is to B, what C is to D (or A/B = C/D which implies an identity between B and D). For example, the petals of the buttercup (5) and the lily petals (3) are in the same ratio as the two successive numbers in the sequence, the largest (5) divided by the smallest (3) leads to the first decimal place, 5/3 = 1,666…

[54] It is no coincidence that the book by the notorious British theoretical physicist Stephen Hawking (1942–2018), which brings together the greatest mathematical texts in history, is entitled *God Created the Integer: The Mathematical Breakthroughs That Changed History*. Hawking, 2005. Reprint 2006. London: Penguin.

Even today, the applications of the golden ratio are found in everyday life[55]. It is part of the language of images, in illustration as well as in photography[56] or in comic strips[57].
To conclude on this point, geometry establishes a link between science and art. This raises the question of the choice of an aesthetic criterion that would be universal and defined *a priori*. This science of demonstrations and proof highlights another and more general question. It is that of the relationship between text and image (or image and text). The term 'image' is here defined in a broad sense, which can mean geometric figures, drawings, or even photographic images associated with a text. These two issues are clarified in the following paragraphs.

2.2.3 The question of aesthetics in Art

Can a universal aesthetic criterion be defined as an undemonstrable axiom? For Emmanuel Kant, in the *Critique of the Faculty of Judgment* (1790), no aesthetic concept of Beauty can be formalised. Beauty is a qualitative and singular notion. Never objective, it is always subjective. According to this thesis, the judgement of taste should not be discussed, since no proof of aestheticism can be provided. However, in mathematics, there is no hesitation in talking about the beauty of a reasoning. This undoubtedly requires understanding the reasoning to conceive its beauty. This discussion on Art and its aesthetic criteria has reappeared in contemporary art. Marcel Duchamp (1887–1968), for example, defined an abstract and original concept in painting: the 'ready-made' (Mink, 2016). With this concept, a bottle holder, a porcelain urinal signed 'R. Mutt', called Fontaine (1917), a snow shovel, can become a work of art if the artist signs it and places it in a certain context. The judgement on the beauty of the artwork or object is based on an 'ideal' relationship between the artist and the spectator. It is up to the viewer of the artwork to perceive what he desires to observe.
Without having to decide here these questions on the possibility of establishing an 'ideal' and aesthetic image/text ratio, it can be observed that Lewis Carroll, such as photographers today, uses geometry in his photographs, with its lines of force and symmetry. His passion for the language of photography is another aspect of the visual language used by Lewis Carroll. The view of the photographer influences the image/text ratio. For instance, it is with the eye of a photographer that he decided on the size and exact locations of the images in the texts, as well as the distance between the words of the text and the illustration[58]. According to my interpretation, it is not only a criterion of aesthetics that must be taken into account here but also a criterion of understanding.

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[55] Most credit cards measure 86 by 54 mm, a rectangle of about 8 by 5. Some playing cards are similar: 'bridge' format: 88 mm/57 mm = 1.543, 'tarot de Marseille' format: 112 mm/61 mm = 1.886, etc.
[56] The standard formats of photography are close to the golden ratio: 13 x 21 cm, 18 x 30 cm, 24 x 39 cm, ratios of about 1.6.
[57] Hergé's work makes great use of the golden ratio in *Tintin.* Example: in *The Crab with Golden Claws*, Captain Haddok's bottle explodes at the golden point (planche 35, case 5). Hergé, 1958. *The Adventures of Tintin. The Crab with Golden Claws.* London: Methuen Children's Books.
[58] Bury, L., L. Gasquet and M. Garrigou-Lagrange, 2019. Lewis Carroll au pays des mystères. Paris: *Papiers, la revue de France Culture*, n° 29, pp. 141-149, and Gattégno (1974, pp. 102–106).

When Lewis Carroll measures the distance between the words of a text and the drawing that illustrates them, he establishes a quantitative ratio of the same nature as the golden ratio. These quantitative measures are especially necessary, such as seen in 'The Logical Spring' pop-up where Alice grows up and then shrinks. For the visual mechanism to operate, the two drawings and the relevant text must be associated. In contrast to the original design by Lewis Carroll's hand[59], this association 'image-text' is not always carried out in the modern illustrations of *Alice's Adventures in Wonderland*, as shown in the case study below (point 2.6). However, when the text and the image are too far apart, the reasoning which uses a logical principle (here the *modus ponens*) can no longer be highlighted. As a result, the theatrical spring of the story, which is supposed to be illustrated, no longer plays its role. This observation led me to take a closer look at the image/text ratio.

2.3 The image/text ratio

The image/text ratio has already been the subject of several studies (Escarpit and Godfrey, 2008, pp. 272–311).

2.3.1 Mapping of the classical image/text ratio

The classical ratio determines a relationship between two languages: that of the images (noted 'l') and that of the texts noted 'L'. Hence three possibilities summarised in the following table**:**

TABLE 37

THE CLASSICAL IMAGE/TEXT RATIO

Three possibilities:

1° 'l identical to L', $l \approx L$. The image (l) replicates the text (L). This is a redundancy criterion. The image has a descriptive role. For example, an illustrated dictionary will establish a visual link with the words. In pedagogy, redundancy facilitates memorisation.

2° 'the image says less than the text', $l < L$. It illuminates the text. It is an aesthetic criterion. It highlights the text and encourages people to read it. It has a rhetorical function that is not directly useful in comprehending the text.

3° 'the image says more than the text', $l > L$. It goes beyond the text and completes it. It is a criterion of complementarity, conjunctions, and narrative continuity. It brings out elements that do not appear in the text. It makes it easier to read. It can introduce an emotion, a sensitivity that is difficult to decipher in words.

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[59] In *Alice's Adventures Under Ground* (1864), the handwritten text and the corresponding drawings are brought together: pp. 10–11, pp. 61–62.

This third case where 'the image says more than the text' is more problematic. It introduces three new criteria: divergence, contradiction, and autonomy.

TABLE 38

WHEN IMAGES SAY MORE THAN WORDS

Three possibilities:

3a. Divergent images offer a different interpretation of the text. This disrupts reading in the case of young children. Unless it is a question of encouraging them to perceive other reading routes (Escarpit and Godfrey, 2008, p. 289).

3b. Conflicting images, called disjunctive, may be voluntary, such as in humorous cartoons or in Magritte's two notorious paintings The Treachery of Images (1929): 'This is not a pipe' and 'This is not an apple' (circa 1964) while the drawings represent a pipe and an apple. Involuntary, contradiction disturbs reading and creates paradoxes. It can harm text and images.

3c. Stand-alone images tell stories independent of the text. This produces an ethical problem for the author. If it is voluntary, the reader will have to discover the links between the stories. If it is unintentional, nonetheless, reading and understanding the text is disrupted by this diversion.

These different image/text ratios reflect the human relationships between the author, the illustrator, and the publisher who represents the public (Salisbury, April 2018, p. 64)[60]. Conflicts are likely to be even greater if the interpretations of text and images are divergent, contradictory or autonomous (such as in points 3a, 3b or 3c above). The complex relationships that existed between Lewis Carroll and his illustrators are an example (Gattégno, 1974, pp. 102–106). This ratio highlights several issues. Is there an 'ideal' L/l ratio such as the golden ratio in Art? What are the criteria for making a quantitative and qualitative judgement on this ratio? How to conceive this ratio when it concerns illustrating a mathematical or logical demonstration? This is where the need to introduce a 'third dimension' to the classic image/text ratio arises.

2.3.2 Using the image/text ratio to illustrate reasoning

The idea is to take into account the context and the objective in which the image/text ratio is established. In practice, this means here differentiating in discourse or argumentation two types of logic: rhetoric and dialectic, and within dialectic to differentiate three main types of reasoning: by deduction, induction and experimentation, by analogy or metaphor.

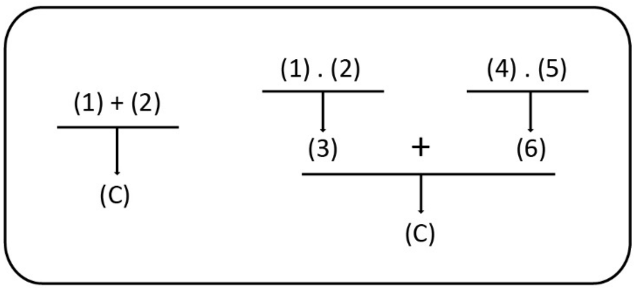
[60] For examples of 'rocky marriages' between author and artist: Salisbury, M., April 2018. The Art of Collaboration. *Literary Review*, p. 64.

TABLE 39

DIALECTICS VS RHETORIC

In rhetoric, argumentation seeks to convince an audience of the correctness of a point of view, a thesis or an opinion by any means. For example, in Plato's Gorgias two theses conflict. According to the sophist Gorgias, the art of good speech and persuasion is the best of all the arts. While Socrates denounces rhetoric as an art of lying. We know Plato was hostile to rhetoric and illusionism (J. de Romilly, 2019, pp. 53-58). To establish a rhetorical approach opposed lying, it was necessary to wait for authors such as Stephen E. Toulmin (1958) and Chaïm-Perelman (1958). They highlighted the dimension of logical reasoning in argumentation. For those authors, introducing an ethical criterion into the discourse consists of justifying one's opinions according to logical rules. For example, an ethical criterion is to admit being wrong when the truth comes out. (Laramée, H. et al., 2009, p. 14.) If the aim was to illustrate the reasoning processes of rhetoric – which is not the objective here – the image/text ratio would require taking into account the techniques of rhetoric. Among these techniques there is the personal attack, the *ad hominem* argument which consists in opposing one's own words to the opponent, hiding the truth and not communicating everything. The most important is probably the argument of authority. It is this argument that Lewis Carroll emphasises in the discussion between the King, the Executioner and the Queen, when the Queen says: 'Sentence first – verdict afterwards.'

In dialectics (in the classical sense of logic and not dialectics in Hegel's thesis[61]), the text takes the form of a demonstration looking for truth and proof. For example, 'If all A is B and all B is C, it necessarily follows that all A is C'. The conclusion is imposed according to the fundamental law of categorical syllogism theory, called the principle of *the dictum de omni-dictum de nullo* (Thibaudeau, 2006, p. 720)[62]. Reasoning can be complex, as in the case of compound syllogisms[63] with several connectors (And, or, etc.), or in syllogisms called sorites which contain a large number of arguments (premises).



Simple and complex reasoning
(Here, two and six arguments connected by And noted '.'; Or noted '+')

[61] For Friedrich Hegel (1770–1834) 'dialectics' is the interweaving of the 'thesis' and 'antithesis' that goes beyond contradiction in 'synthesis'. Hence, the negation of the negation does not give the affirmation back, but something else. (Ellul, 2003. *La pensée marxiste*. Paris: La table ronde). This is not the logic that Lewis Carroll is referring to. In classical logic, the negation of the negation gives back the affirmation: no (not-p) = p.

[62] Thibaudeau, V., 2006, p. 720. Original text in French: 'Quand un terme est attribué universellement à un sujet, il doit nécessairement s'attribuer à tout ce qui est contenu dans l'extension de ce sujet (*dictum de omni*). Quand un terme est nié universellement d'un sujet, il doit nécessairement être nié de tout ce qui est contenu dans l'extension de ce sujet (*dictum de nullo*).'

[63] Laramée, H. et al., 2009, pp. 50–55, and detailed structure of the various types of compound enunciations in Thibaudeau (2006, pp. 553–558 and exercises, pp. 559–560).

As Aristotle observes in his *Sophistical Refutations*[64], some reasoning is unintentionally false, but can be intentionally wrong, in order to deceive interlocutors in a discourse such as do the Sophists that Aristotle fights. Consequently, the image/text ratio must be able to illustrate both correct and fallacious reasoning and help to distinguish between rhetoric and dialectic (logic).

2.4 Illustrating fallacies and paradoxes

Lewis Carroll, like Aristotle, considered the art of reasoning as a means of detecting false reasoning, called sophisms. The most dangerous are paralogisms, i.e. reasoning that gives the impression of being true when it is false. Paradoxes are also dangerous traps for the mind because it is difficult to find a solution to escape from them. This is why, according to Carroll, children must be taught at a very early age about false reasoning, paralogisms and paradoxes. They have several origins. *On Sophistical Refutations*, Aristotle establishes a classification, still used today, in which he distinguishes five types of fallacious reasoning: refutation, errors, paradoxes, solecisms (phrases that transgress the rules of grammar) and verbiage (to monopolise the floor with nothing essential and even false reasoning). He examines in detail the paralogisms. He strongly condemns sophists, such as Corax of Syracuse (6th- 5thcentury B.C.), who make great use of them for their own benefit, independently of any search for truth. This is even, according to Aristotle, the definition of a sophist when he writes[65]: 'the art of the sophist is the semblance of wisdom without the reality, and the sophist is one who makes money from apparent but unreal wisdom'. Following Aristotle, Lewis Carroll in his introduction to *Symbolic logic* is very explicit: learning logic is not a question of teaching children nonsense, but on the contrary, to help them to detect fallacious reasoning that they may find 'in books, in newspapers, in speeches, and even in sermons'. (Introduction, 1896, p. xvii.) Since the aim here is not to illustrate rhetorical texts, the focus will be on illustrating the two points of Aristotle: firstly, the involuntary reasoning error and secondly the paradox that present a greater complexity of understanding.

2.4.1 First objective: To illustrate the error of involuntary reasoning

In a deductive argumentation where certain premises are stated and a conclusion other than what has been stated follows, the involuntary error may have two origins: the arguments used and the reasoning itself.

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[64] In Aristotle's *Organon*, the *Sophistical Refutations* complete the *Topics*. False reasoning, including that of the Sophists, is analysed and criticised here. Barnes, 1998; McKeon, 2001, *Topica,* p. 188, *On Sophistical Refutations*, p. 208; Tricot, 2007: *Organon VI*, Paris: J. Vrin.

[65] Aristotle, 384-322 BC. Translated and introduction by C.D.C. Reeve*,* edited by R. McKeon, *2001. The basic works of Aristotle*. *On Sophistical Refutations*. New York: Modern Library Paperback Edition, p. 209.

Arguments (premises) may be only probable, contradictory, or appear to be true when in reality they are false, and so on. The question to be considered here is that of the truth of the premises and the conclusion. The important point is that formal logic is mainly concerned with the *validity* of the reasoning and not in the *truth* of the premises. Reasoning can be valid, invalid or indeterminate. It depends on the correct or incorrect way in which the axioms, definitions and rules are used. For the illustrator, the aim is to show and make children understand the difference between these two concepts 'truth' and 'validity': i.e., on the one hand, the *truth* of the premises and the conclusion and, on the other hand, the *validity* of the reasoning that allows the conclusion to be inferred from the premises. In other words, it is the principle of deductive reasoning itself that needs to be illustrated. To this end, I have created several jigsaw puzzle games.

PERSONAL ILLUSTRATIONS

24 VALID SYLLOGISMS

Puzzles 1 to 3 start by making children aware that not all reasonings are necessarily valid, and, consequently in a discourse, in a syllogism, there are valid and invalid conclusions. Several examples are illustrated in the puzzles but also in Games 4 to 7. If the puzzles 1 to 3 simply show that certain reasoning can be wrong, the games 4 to 7 and the Booklets illustrate the reasons why they can be wrong.

PERSONAL ILLUSTRATIONS

2.4.2 Second objective: Illustrating paradoxes

The most challenging question is certainly that of paradoxes. They can go so far as to endanger the principle of deduction itself. This is what Lewis Carroll highlights through his stories of paradoxes. He shows that it is necessary to clearly distinguish between the premises and the specific rules of deduction. Taking the deduction rules into account has important consequences for the understanding of what needs to be illustrated and the difficulties involved. The aim is to make children understand these rules through play and visual arts. For example, nothing universal can be concluded from a particular case, just as nothing can be concluded from two negative[66] or contradictory premises (which is examined in Game 7). Since antiquity, paradox has been a short story whose aim is to denounce nonsense. As any story or tale, it can be illustrated. The question becomes more difficult if one wants to show the mechanism of the paradox. For it is generally based on the logical principle of contradiction, as is the case in Carroll's Cheshire cat paradox which I illustrated by means of a pop-up.

[66] In symbolic form, this classical Aristotelian rule: 'from two negative premises no conclusion can be draw' indicates, for example, that the following syllogisms are invalid: No M is P and No S is M, such as No M is P and some S is not M are invalid reasoning.

The American logician W. V. Quine (1908–2000), one of the leading representatives of analytical philosophy, classifies paradoxes into three categories: veridical, fallacious and antinomic (Quine, 1976). This leads to illustrate three very different types of paradoxes, some of which have been known since antiquity, and nowadays serve as a reference in science to highlight paradoxical phenomena. The most famous example is Russell's Barber's Paradox concerning the mathematical theory of sets.

TABLE 40

THREE CATEGORIES OF PARADOXES ACCORDING TO QUINE'S CLASSIFICATION

- The 'veridical' or truth-telling paradox leads to a surprising conclusion, whereas the reasoning seems perfectly correct. However, the surprise disappears when you understand the reason for the surprise.

- The 'falsidical' paradox is surprising. Astonishment is based on an error of reasoning or judgement. It can be explained by the lack of knowledge of certain philosophical, mathematical and scientific truths, or by charlatanism, which exists in science as well as in other fields.

- The third category of paradox is the 'antinomic' paradox. It requires the renunciation of certain generally accepted truths. This means making discoveries. What was impossible at once becomes possible at another. When the paradox is resolved, it returns to the category of syllogisms whose conclusion can be true, false or undecidable. As Quine points out, there was a time when the doctrine of the Earth's rotation around the Sun was called the Copernican paradox. Similarly, some of Newton's doctrines on gravity have been challenged by Einstein's theory of relativity[67].

If the Cheshire cat paradox is presented implicitly, it is explicitly that Lewis Carroll submits to his audience paradoxes such as *What the Tortoise Said to Achilles* (1894) or *The Barbershop Paradox,* first titled '*A Logical Paradox'* (1894). To discover their origin, to understand the problems they pose and to be able to illustrate their logical basis, I examine below the three categories of paradoxes defined by Quine (1976). In game 7.1 players are invited to create their own story and will be able to draw inspiration from the short stories of paradoxes known since antiquity. The pedagogical advantage of these thought-provoking short stories is that they are written in everyday language, understandable by all, what can then be put in symbolic form. In this manner the image/text relationship must permit visualisation of correct and fallacious reasoning by means of specific tools (games design, pop-up games, cards, illustrated counters, etc.) and help to distinguish between rhetoric and dialectic (logic). In this case the term 'illustration' will be extended to a broader concept of meaning: that of 'visualisation' or 'game design'. The very nature and usefulness of a concept is that it can be extended (as will be seen later in Chapter IV).

[67] For Newton the space is empty, the galaxies and the stars influence each other whereas for Einstein the space is constituted of a four-dimensional fabric: the 'space-time' in which it is the galaxies, the stars and the planets that deform it.

1. The Barber's paradox

This paradox was conceived in 1918 by the English logician and philosopher Bertrand Russell (1872–1970). It is a Barber story as told by Lewis Carroll in the Barbershop Paradox, but the two paradoxes of Russell and Carroll are different. They have several possible classifications and interpretations.

TABLE 41

THE STORY OF RUSSELL'S BARBER

The mayor of a village orders his barber to shave all the inhabitants of the village who do not shave themselves. However, the barber who is a resident of the village cannot respect this rule. Because:

– If he shaves himself, he violates the mayor's rule, since he can only shave residents who do not shave themselves;

– If he does not shave himself, he also breaks the rule, since he must shave all the men in the village who do not shave themselves.

For Olin (2003, p. 13.), the Barber's paradox is a perfect example of a veridical paradox. The conclusion is contradictory, 'the barber cuts his own hair if and only if he does not cut his own hair'. Cheng (2019, p. 159) highlights here the principle of contradiction as a dilemma in the following way[68] :

TABLE 42

THE BARBER'S DILEMMA

'If person A shaves person A, then the barber doesn't shave person A.

If person A does not shave person A, then the barber shaves person A.'

Because A represents any person in the town, the letter A can be replaced by 'the barber'.
The two statements become:

'If the barber shaves the barber, then the barber doesn't shave the barber.

If the barber does not shave the barber, then the barber shaves the barber.'

Cheng concludes: 'Each of these statements produces a contradiction'.

[68] Cheng, E., 2019. *The Art of Logic. How to make sense in a World that doesn't*. 1st ed. 2018. London: Profile Books Ltd.

In *Paradoxes*, the British philosopher Sainsbury (1995) simply states, ‘that such a barber cannot exist’. Because with such a contradiction, there can be no village that can have such a barber. For Russell, the barber’s paradox is an antinomy that poses a serious scientific problem and for this reason this paradox is also called *Russell’s antinomy*. Its importance is undeniable. It will be at the origin of the *Crisis in the Foundations of Mathematics* at the turn of the 20th century. It illustrates in a didactic way a more complex paradox on set theory: ‘Do all sets that are not members of themselves belong to each other?’ This sentence highlights a contradiction. A set cannot be both an element of itself and not an element of itself, nor can it be the set of all sets and not the set of all sets that are larger than it (Vidal-Rosset, 2004, p. 15). To avoid the paradox, Russell’s type theory, formulated in 1903, states as a rule that ‘we must give up talking about the set of all sets’. It only allows reference to all sets of a certain type. Yet, this paradox, such as many others, continues to haunt the nights of logicians looking for a more satisfactory solution[69]. The truth is that it is difficult to emerge from an antinomic paradox.

2. The paradox of Achilles and the Tortoise

This paradox is attributed to Zeno of Elea (circa 490-430 B.C.). It is originally a philosophical paradox whose premises deny the existence of the movement. It is to this ancient paradox that Lewis Carroll refers in *What the Tortoise Said to Achilles* (1894). Here is the story.

TABLE 43

THE ORIGINAL STORY OF ACHILLES AND THE TORTOISE

Zeno of Elea claims that in a race between Achilles, the legendary hero of the Trojan War, and a tortoise, Achilles will never catch the tortoise.

Here is the argument. Achilles, a fast runner, allowed the tortoise to start one step ahead. With a leap, he jumps to the place where the tortoise is. In order to catch up he takes another strike and at the same time the tortoise has advanced further.

Conclusion: Achilles will never be able to catch the tortoise.

Some authors have mathematically tried to demonstrate that this story is a false reasoning.

[69] Graham Priest – 6. *Paradoxes. Lecture 6. Romanae Disputationes*, 20 Feb. 2017. CUNY Graduate Center (NY): University of Melbourne. [online] Available at: <https://www.youtube.com/watch?v=BlOyKhvFK40> [Accessed 16 June 2020].

TABLE 44

THE ACHILLES PARADOX IN MATHEMATICAL FORM

As reported by Hayden and Picard (2009), some authors[70] interpret the reasoning by an infinite series of diminishing distances: 1/2 + 1/4 + 1/8 + 1/16… +… + 1/2n + 1/2n+1… where n tends to infinity ($2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc.). Achilles advances one metre, then half a meter, then a quarter of a meter, and so on. Hence this infinite series of terms. This series of numbers is often illustrated by the drawing of half circles whose diameter narrows with each Aristotle's jump. Then these authors use a mathematical demonstration that this series converges towards a limit equal to 1. For them the paradox is solved. Achilles can catch the tortoise.

Quine (1976) classifies the 'Achilles paradox' in the category of falsidical paradoxes[71].
However, one may wonder, and ask the question, can this paradox be truly resolved by a sequence of numbers that tends towards a finite limit, equal to 1? One can doubt it. If one refers to the mathematical definition of a limit, then this analogy with a mathematical sequence paradoxically gives reason to Zeno of Elea: Achilles will never catch up with the tortoise. Because if the terms of a convergent sequence can become as close as one wants to a finite limit (say L and write: $\lim_{n \to +\infty} x_n = L.$), by definition of a limit, no term can reach it. When L tends towards 1, this does not mean that L is equal to 1. This shows above all the weakness of the reasoning by analogy which consists here in interpreting Zeno's text as a sequence of numbers. As Thibaudeau (2006, p. 787) points out, reasoning by analogy can be very effective in convincing, but it is 'superficial and less rigorous than deductive reasoning'. Lewis Carroll may have known this mathematical interpretation of the paradox of Zeno and to close the debate, he may have decided to put Achilles on the Tortoise's back. These are the first words of his tale: 'Achilles had overtaken the Tortoise, and had seated himself comfortably on its back.' That means L = 1. This raises the more general question of understanding and interpreting a text before being able to illustrate it.
In his 'Solutions of Classical Puzzles', Lewis Carroll tells the story differently and concludes instead that Achilles will overtake the Tortoise. This leads to two different solutions to the same problem.

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[70] Hayden, G. and Picard, M., 2009. *This book does not exist. Adventures in the paradoxical.* New York: Fall River Press Publisher. Translated into French by C. Nioche, 2013. *Ce livre n'existe pas. Paradoxes, énigmes mathématiques et énigmes philosophiques*. Paris: Marabout. Hachette, pp. 114–119.
[71] Quine uses here the expression *falsidicus, which* he found in particular in Plautus (c. 254-184 B.C.), a Latin comic author who would have influenced, among others, Shakespeare and Molière.

TABLE 45

LEWIS CARROLL'S VERSION OF ACHILLES AND THE TORTOISE

Lewis Carroll's story:

The Tortoise: 'That beautiful First Proposition of Euclid!' the tortoise murmured dreamily. 'You admire Euclid?'
Achilles: 'Passionately!'…

(A) Things that are equal to the same are equal to each other.
(B) The two sides of this Triangle are things that are equal to the same.
(Z) The two sides of this Triangle are equal to each other.'

Bartley (2017, Appendix C, Editor's note, p. 466) sums up: 'The Tortoise points out to Achilles that a person might refuse to accept the conclusion (Z) on two different grounds: He might deny the truth of the premises; or, accepting the premises as true, he might deny *the validity of the inference* from the conclusion.'

By confusing the two concepts, one from the field of language (the implication: If … then), the other from metalanguage (the rules of deduction: therefore, from the premises one can conclude that…), the Tortoise forces Achilles to the logical absurdity of infinite regression:

(C) If A and B are true, Z must be true.
(D) If A and B and C are true, Z must be true.

And so on, *ad infinitum*.

How can one be sure that it is possible to deduce that conclusion Z is valid? From deduction to deduction Achilles must admit a series of hypothetical propositions that carry on to infinity until exhausted.

(Carroll, L., 1894. *What the Tortoise Said to Achilles.* Reprint 2006. London: Wordsworth Editions, pp. 1179–1182, and Bartley, 1977, Lewis Carroll's text, pp. 431–434, and Appendix C, Editor's note pp. 466–470).

In *What the Tortoise Said to Achilles* (Mind magazine, 1894), Carroll humorously-distorts the paradox of Zeno of Elea. In his tale, he is interested in infinitely divergent sequences and not in those that converge towards a finite limit.

TABLE 46

ACHILLES AND THE TORTOISE: LEWIS CARROLL'S SOLUTION:

Excerpt from Lewis Carroll's Achilles and the Tortoise; Bartley (1977, p. 426 and pp. 438–439.):
"The legend runs as follows: Achilles and the Tortoise were to run a race on a circular course; and as it was known that Achilles could run ten times as fast as the Tortoise, the latter was allowed 100 yards' start.

By the time Achilles had run the 100 yards, the Tortoise would have got 10 yards further; and, by the time he had run those 10 yards, it would have got a yard further; and so on forever. '
Carroll's conclusion: 'This is a mathematical Fallacy, and involves the false assumption that a series of distances, infinite as to number, is also infinite as to total length. Here the assumption is that:

$(111 + 1/10 + 1/10^2 + 1/10^3 + \&c.)$ of a mile, where the number of terms can be greater than any assigned number, can be made greater than any assigned length. But the above series is the circulating decimal 111.1 which as the Reader probably knows can never reach the limit $111^{1/9}$. Hence, by the time Achilles has run $111^{1/9}$ yards, he must necessarily have overtaken the Tortoise.'

Lewis Carroll's new interpretation of an ancient paradox raises the more general question of understanding and interpreting a text before being able to illustrate it. The illustration of the image/text ratio will not give the same result depending on whether it is considered that Achilles cannot overtake the tortoise, is sitting on the tortoise's back or in the race overtakes the tortoise. Lewis Carroll's paradox will capture Bertrand Russell's attention. The latter quotes him in his *Principles of mathematics* (1910*)* when he discusses the notions of implication, inference and recursiveness ('if A then B', 'A Implies B', 'B implies C,' etc.). This shows that the matter of paradoxes is a serious issue and difficulty. It is this paradox that challenges the principle of deduction itself and makes clear the distinction between rules and premises, as Gattégno and Coumet (1996) point out in their comments on Lewis Carroll's paradox. It is the whole problem of deduction, which is the very principle of mathematics, that is called into question by Lewis Carroll's paradox: A = B, B = C, C = D, D = E, therefore A = E. How to prove that A = E? In short, the problem of deduction is challenged by a single word: 'therefore' (thus, so, consequently, etc.). The correct application of the *modus ponens* will allow a means to get out of the paradox. In the Carrollian story, one can recognise the application of the *modus ponens* used in the paradox of Eubulides or in the paradox of the grain of sand, which bears the name of sorite, coming from the translation of 'sõros', 'heap' in ancient Greek.

TABLE 47

THE SORITE PARADOX

Here is the story.

If 10,000 grains of sand make a heap of sand, then 9999 grains make a heap of sand. However, 10,000 grains of sand make a heap of sand. So, 9999 grains of sand make a heap of sand (application of the *ponens modus*).
If 9999 grains of sand make a heap of sand, then 9998 grains make a heap of sand. However, 9999 grains of sand make a heap of sand.
So, 9998 grains of sand make a heap of sand, and so on…
Conclusion, a single grain of sand forms a heap, and, if we continue the reasoning, 0 grain of sand makes a heap of sand. Which is an illustration of nonsense.

This paradox is easily resolved here by defining more precisely what is called a pile of sand[72]. However, it raises another crucial question: the confusion between premises and rules, that is between language and metalanguage. This confusion, which produces a paradox, was highlighted by Eubulides of Miletus, the Greek philosopher of the Megarian school, born at the end of the 5th century BC.

3. The Liar's Paradox

A paradox can be both truthful and antinomic. It surprises, but states the truth, the proof of which is at the limit of the paradox. This is the case of *the liar's paradox*, known to the ancients as the *pseudomenon*[73]. It simply says, 'I'm lying.'

[72] Other example: Graham Priest – 6. *Paradoxes. Lecture 6. Romanae Disputationes*, 20 Feb. 2017. CUNY Graduate Center (NY): University of Melbourne. [Online] Available at: <https://www.youtube.com/watch?v=BlOyKhvFK40> [Accessed 16 June 2020].

[73] 'Pseudomenon': from the Greek: pseudo meaning false and menon meaning to deceive.

TABLE 48

EUBULIDES' PARADOX

Eubulides says, 'I lie.'

–If he lies by saying 'I'm lying' then he's not lying, he's telling the truth;

–But if he tells the truth by saying 'I'm lying' then he's lying, so he's not lying, because he's telling the truth.

How can he, without contradiction, both lie and not lie?

The paradox comes down to an infernal cycle. If it is true, it is false, if it is false, it is true, and so on. That's enough to lose your mind as does Lewis Carroll's Cheshire Cat.

4. In Pseudomenos[74], Lewis Carroll takes up this paradox in form: 'If a man says,' I am telling a lie,' and speaks truly, he is telling a lie, and therefore speaks falsely, but if he speaks falsely, he is not telling a lie, and therefore speaks truly.'
All these types of paradoxes appear when a sentence speaks for itself. Such as Magritte's paradox seen above: 'this is not a pipe', this creates an ambiguity between language and metalanguage. To avoid the paradox, it is necessary to distinguish the two levels of language. This is the solution recommended by the logician and mathematician Alfred Tarski (1901–1983)[75]. As shown in Games 4 to 7 and the Booklets, the distinction between the two languages helps to avoid paradoxes. In the compound syllogisms, the Stoics highlighted the need to use in reasoning that they call the five indemonstrables of which the *modus ponens* is a part. The application of the *modus ponens* rule allows Carroll's paradox to be resolved here in the following form.

[74] Extracts from W. W. Bartley III in *Scientific American*, a division of *Nature America, Inc*., July 1972. The manuscript from which extracts have been taken is in the possession of Christ Church, Oxford; Bartley (1977, p. 425 and pp. 434–436).
[75] Tarski, A., 1946. *Introduction to logic and to the methodology of deductive sciences.* New York: Dover Publications, Inc. Reprint 1995. Translated by O. Helmer. New York: Dover Publications, Inc. Translated from English into French by J. Tremblay S.J., 1971, 3rd ed. Paris: Gauthier-Villars.

TABLE 49

APPLICATION OF THE MODUS PONENS

If A and B then Z
and A
and B
and A and B
Therefore, Z

The *modus ponens*, such as the Aristotle's *dictum de omni et nullo* principle, is part of the rules of metalanguage which are neither true nor false, but useful for solving logical problems, even if they are indemonstrable. This makes it possible to accept the term 'therefore'. If A and B and C and D… then Z, and A, B, C, D… are true, one must admit Z, contrary to what the Tortoise says. Achilles was wrong to confuse language with metalanguage.

5. Lewis Carroll's Barbershop Paradox

In the *Barbershop* paradox[76] Lewis Carroll gives an example of a misuse of the principle of reasoning by the absurd known since antiquity under the name of *reductio ad absurdum*. In the *Organon*, Aristotle uses this reasoning to demonstrate the validity of the categorical syllogisms of figures II Baroco and III Bocardo that I illustrated in the puzzles and Booklet 3. Paradoxically, to validate reasoning, the principle of *reductio ad absurdum* is based on the combination of nonsense and contradiction as an antidote to false reasoning. I summarise here the story of Lewis Carroll's Barbershop Paradox and expose why the reasoning is fallacious.

[76] Carroll L., 1894. A Logical Paradox. Oxford: *Mind*, New Series, Vol. 3, No. 11, July, pp. 436–438. Bartley (1977, Appendix A, editor's note, p. 444) writes: 'What is known as Lewis Carroll's "Barber-Shop Paradox" is one of the most curious anomalies of logical controversy during the past eighty years. Eight versions of the puzzle exist.' There was a main disagreement between John Cook Wilson, professor of logic at Oxford, and Lewis Carroll, on the question of the distinction between premises and the rule of inference. Bertrand Russell discussed the paradox in *The Principles of Mathematics* in 1903 and this kind of problem was described by Boole, Jevons, Venn, J.N. Keynes, A. Sidgwick (Bartley, 1977, pp. 444–465). I give here the principle and the solution under the heading 'The Barbershop Paradox'. The paradox is translated into French and commented by J. Gattégno and E. Coumet, 1966. *Logique sans peine*. Reprint, 2006. Paris: Hermann Éditeur, 6th edition, pp. 249–253 and translated by Gattégno et al., 1990. *Lewis Carroll. Œuvres*. Paris: Bibliothèque de la Pléiade, Gallimard, *Les trois coiffeurs,* pp. 1626–1629.

TABLE 50

THE STORY OF LEWIS CARROLL'S BARBERSHOP PARADOX

Three hairdressers work in a Barber shop: Allen, Brown and Carr, but they are not all always present in the store. Carr is a good hairdresser and Uncle Jim wants to be shaved by him. He knows the store is open, and

1st: one of them must be present, and
2nd: Allen never leaves the store without Brown.

The problem: Uncle Joe insists, Carr is surely present. He says he can prove it logically using the principle of *reductio ad absurdum.* Uncle Jim asks for proof.

The problem can be written in the following symbolic form of a hypothetical syllogism:

(1) If Carr is out, then if Allen is out, Brown is in;
(2) If Allen is out, Brown is out.

The question is: can Carr be in?

By designating the names Allen, Brown, Carr with the letters A, B, C and replacing the words "out" by "true" and "in" by "not true", the problem can be written:

(1) If C is true, then if A is true, B is not true;
(2) If A is true, B is true.

The formal question is, can C be true?

In other words, are two hypotheticals of the forms "If A, then B" and 'If A, then not-B" compatible?

According to the principle of *reductio ad absurdum*, to prove a hypothesis is false, Uncle Joe begins by assuming it to be true. He therefore assumes Carr is absent. In this case, if Allen is also absent, Brown must be present (according to the 1st proposition). However, if Allen is absent, Brown is also absent (according to the 2nd proposition). He obtains two contradictory propositions, 'Brown is present' and 'Brown is absent'. These two propositions cannot be true simultaneously. Therefore, the starting assumption 'Carr is absent' is false. Hence his deduction, 'Carr is necessarily present.' However, this reasoning is fallacious. Uncle Jim points out if Allen is present, nothing prevents Carr from being absent, whether Brown is also present or absent. Uncle Joe's reasoning is therefore incomplete. It did not take into account all possible cases.

A more complete demonstration is to use a cross-table presentation listing all acceptable combinations. Wittgenstein generalised this idea in 1920 with the Truth Tables present in his *Tractatus-Logico-Philosophicus*[77].

TABLE 51

THE BARBERSHOP PARADOX OR THE PRINCIPLE OF COMBINING ALL POSSIBLE CASES

The combination principle is to list all possible cases in an exhaustive manner and to take into account the two constraints imposed. Since there are three barbers (n = 3) and only two possibilities for each of them (p = 2) to be present ('in') or out ('out'), this makes a total of n^P possible combinations, i.e. $2^3 = 2 \times 2 \times 2 = 8$ possibilities.

Possibilities	1	2	3	4	5	6	7	8
A: Allen	out	out	out	out	in	in	in	in
B: Brown	out	out	in	in	out	out	in	in
C: Carr	out	in	out	in	out	in	out	in

Problem:

Does Uncle Joe really prove, as he claims, that Uncle Jim's preferred barber will be there when they arrive at the Barbershop?

Answer:

The two constraints imposed are found in the following columns 1, 3 and 4. These options are excluded. Column 1: the three barbers cannot be absent at the same time. Columns 3 and 4: if Allen is out, Brown is with him outside ('out'), knowing that Allen never leaves the store without Brown.

Column 2 shows Uncle Joe's solution. Carr must be present, because if Allen is absent (out), Brown is also absent (out). But Uncle Joe's solution is incomplete. There are two other solutions that he does not mention. Columns 6 and 8 show that Carr can be in the Barbershop (in), if Allen is present (in) whether Brown is outside (out) or inside (in).

In the end, contrary to what Uncle Joe thinks, Carr may be absent in two cases (Col. 5 and 7): when Allen is present whether Brown is present or not.

[77] To put an end to the contradictory opinions of the logicians, and probably to put an end to a badly posed problem as the Pólya method of problem solving (1945) would show, instead of reasoning by the absurd, Lewis Carroll proposed, a very simple method, which consisted of examining all possible combinations. He writes (September 1894): 'There are eight conceivable combinations of A, B, C, with regard to truth and falsity (Bartley, 1977, p. 465): two solutions contain the condition "C is true"; commented also by Ernest Coumet, 1989. *Lewis Carroll, Paris*: Éditions Bouquins, Robert Laffont, vol. 2. p. 756. Lewis Carroll matrix is presented here replacing the words "true" by "out" and "false" by "in".

This Lewis Carroll's paradox shows the importance of considering all possible cases exhaustively. In Game 7, called 'The Robot', I highlight this concept of exhaustive enumeration by using Truth Tables with counters and a game board. Here, the Barber's Paradox shows that reasoning without being totally false can be incomplete. It is this principle, used by Sophists to reveal only part of the truth (or falsity) to convince at all costs which is condemned by Aristotle. In the end, in this barber's story, it is interesting to observe that nothing can be concluded for certain. Carr, Uncle Jim's favourite barber, may or may not be present. To answer the question: Will Carr be present? It is necessary to use the probability calculus. With Blaise Pascal (1623–1662), one of the founders of the mathematical treatment of probabilities, the principle of exhaustive enumeration was to play an important role in the calculation of probabilities and the discovery of uncertain worlds and games of chance.

TABLE 52

PROBABILISTIC INTERPRETATION OF LEWIS CARROLL'S PARADOX

In mathematics, probability theory is the study of phenomena characterised by chance and uncertainty. It is based on the combinatorial theory. In its simplest expression, the probability P(A) of an event A is the ratio between the number of favourable cases for A and the number of possible cases:

$$\text{Probability} = \frac{\text{Number of ways for something to happen}}{\text{Total number of possible outcomes}}$$

By definition, this number is between 0 and 1. The number of possible cases is obtained by an exhaustive enumeration. In the barber's paradox, there are a total of 8 possible cases, 3 of which are excluded (columns 1, 3 and 4). Uncle Joe is right in saying that Carr will be present in 3 cases (columns 2, 6 and 8) and he is wrong in two cases (columns 5 and 7) where Carr may be absent. In terms of probability Carr has a 3 out of 8 chance of being present and 2 out of 8 chance of being absent, i.e. a probability of 0.37 versus 0.25 or 37% chance of being present versus 25% chance of being absent. Despite a higher probability for Carr to be present, it remains low. Taking into account the exclusions: 3 impossibilities out of 8 or 37%, the sum of the probabilities $(3/8 + 2/8 + 3/8)$ is indeed equal to 1 or 100%: $(3{+}2{+}3)/8 = 1$.

This barber's paradox proves reasoning by the absurd is incorrect here, but it does not lead to any certainty as to whether Carr will be present. However, it opens the way to the world of chance and uncertainty, empirical and experimental sciences and other forms of logic that go beyond the framework of this research.

This world of uncertainty or hypothesis[78] is often directly related to the way Venn diagrams are used today (Game 5). In educational games for children, Venn diagrams are mainly used to classify and enumerate subsets of things, and later in their schooling to calculate probabilities. I will give here only an example of illustrations of Venn diagrams applied to the calculation of probabilities. This makes it possible to distinguish two other important concepts, the probability of totally independent events and the probability of dependent events. Game 5 shows that the Venn diagrams can be used differently than to group numbers.

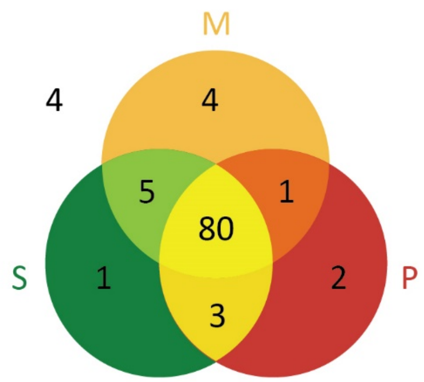
TABLE 53

A QUIZ FOR CHILDREN

The story. 100 students tasted three smoothies S, M, P and their mixture.
The results are as follows[79]:

89 like smoothie S,
90 like smoothie M
86 like smoothie P
83 like smoothies S and P
85 like S and M smoothies
81 like smoothies P and M
80 like smoothie S, P and M
First question. Draw a Venn diagram.

Answer:



Second question. Calculate the following probabilities.

(a) Prob (like none of the three smoothies S, M, P)
(b) Prob (like P but not S)
(c) Prob (like any smoothie except M)
(d) Prob (like exactly two of the three smoothies)
(e) Prob (like C given that a student likes P)

Answers: a = 4/100 (or 4%); b = 3/100; c = 6/100; d = 9/100; e = 81/86[80].

[78] This world of hypothesis is that of the physical and experimental sciences. As the mathematician and physicist Henri Poincaré (1854–1912) himself wrote in *Science and Hypothesis*, science, in the sense of natural and physical sciences, is based on hypotheses. Poincaré, H., 1902. *La science et l'Hypothèse*. Paris: Flammarion, 1902. Reprint with a biography of the author and a preface by Jules Vuillemin, 1968. Paris: Flammarion. Translated into English by W. J. Greenstreet, 2014. *Science and Hypothesis*. Scotts Valley, California: CreateSpace Independent Publishing Platform.

[79] *Probability Venn Diagram Example: ExamSolutions*, 2011. YouTube Channel. [online] Available at: <https://www.youtube.com/watch?v=gaEvsvHb4kk> [Accessed 19 June 2020]. The example of 'a wine tasting' is replaced here by a smoothie tasting for children. Only the numbers and answers are kept here.

[80] Note: (e) is a conditional probability whose formula is Prob (M/P) = Prob (M and P)/Prob(P) =[81/100]/[86/100] = 81/86.

From this analysis of paradoxes, I draw five main conclusions for illustration.

2.4.3. Five conclusions to illustrate syllogisms and paradoxes

First conclusion: the link between storytelling and philosophical reasoning has been established since antiquity. Logicians and mathematicians such as Russell or Lewis Carroll followed this tradition to highlight paradoxes through storytelling and to solve logical and mathematical syllogisms. I take up this idea of linking pure science and storytelling in Game 7. However, here, it is the player himself who can design his own story to solve syllogisms, dilemmas, contradictions or paradoxes by means of Truth Tables. To do this, he has at his disposal counters to help him to build a story and solve the puzzles, a game board to use the Truth Tables and an instruction manual (Booklet 7).
Second conclusion. In paradoxes, there is an implicit link between two principles: nonsense and contradiction, the first serving as an antidote to the second. By making this relationship explicit, the categories of paradoxes can be redefined according to these two principles.

TABLE 54

THE THREE PARADOXES FORMULAE

1. The veridical paradox = nonsense + a contradiction that is usually at the conclusion level.

2. The fasidical paradox = nonsense + a contradiction that comes from reasoning.

3. The antinomic paradox = nonsense + a contradiction between the premises and conclusions that are valid at one time, but false at another.

Third conclusion: if one wants to avoid falling into the trap of paradoxes, one solution is to distinguish between premises and rules, i.e. language and metalanguage. This more theoretical question is dealt with in the second part (Chapter IV).

Fourth conclusion: as a consequence of the first three conclusions, the text and the illustration, and therefore the image/text ratio, need to take into account the two languages which, in terms of semantics and syntax, are two distinct forms of reasoning.

Fifth conclusion: this approach allows me to visually reinterpret the Carrollian nonsense and to consider its illustration, as shown in the following point.

2.5 Reinterpretation of the Carrollian nonsense and its illustration

Many commentators[81] have seen the "nonsense" as a central point in Lewis Carroll's work. It can take many forms as in puns (*Word links*, *Doublet*s, *Lanrick*, *Mischmasch*, *Syzygies*), and/or be strange and surprising (a headless cat in the Cheshire cat paradox).

2.5.1 The humorous nonsense

It is indeed the case that, nonsense is part of the common language that Lewis Carroll uses in his tales. I sum up the humorous nonsense with the following formula:

Formula I. The humorous nonsense:

Carrollian nonsense = contradiction and opposition in words, sentences and conclusions (puns and conclusions that surprise).

The nonsense can also be a fiction depending on the psychological or logical interpretation given to it. In *Alice's Adventures in Wonderland*, Alice can grow ('opening out like the largest telescope that ever was!') or shrink ('shutting up like a telescope') by drinking a magic potion or eating the edge of a mushroom. The psychological interpretation sees a real little girl, Alice Liddell, growing up. The logical interpretation made here is the use of a theatrical spring based on the mechanics of the *modus ponens*. It is this aspect of nonsense that I illustrated in the first pop-up (Preliminary Game 1) called, for this reason, *The Logical Spring*. These two approaches are not incompatible, the second, which allows a visual interpretation of nonsense and not only a textual one, can even reinforce the first.

2.5.2 The 'logical nonsense'

However, as some critics of Carroll pointed out at the time, one could see in nonsense, foolishness to entertain children. But that would be a paradox. The logic to which Carroll constantly refers attaches particular importance to the fight against silliness and fallacies. He preaches nonsense to make children discover its absurdity. It is quite simply the logical use of reasoning by the absurd, from Latin, the *reductio ad absurdum*: to preach the false to discover the truth. To prove a hypothesis is false, one begins by assuming it is true. That is where the nonsense is. If the conclusion is absurd, we conclude the starting assumption was wrong. Conversely, to prove a hypothesis is true, it is first assumed to be false. Hence the nonsense.

[81] Gardner, M. 1960. *Lewis Carroll. The annotated Alice. The definitive edition*. USA: Clarkson N. Potter Inc. Reprint 2001. London: Penguin Books, p. 327, selected references: 'On Nonsense'.

If the conclusion is absurd, we deduce there was no reason to assume it to be false. Thus, nonsense produces, first, a comic situation or reflection, secondly a reasoning that allows the restoration of the truth. In brief, nonsense is an antidote to sophisms, lies and ignorance. The formula II of which I call 'logical nonsense' is composed of two contradictory and powerful ingredients, nonsense and contradiction.

Formula II (the antidote). The "logical nonsense":

Carrollian logical nonsense = "Carrollian nonsense" + reasonin by the absurd

2.5.3 The general formula of the 'logical nonsense'

The *reductio ad absurdum*, that is, a 'reduction to absurdity' may seem stranger than fiction. However, reasoning by the absurd - when used properly - is based on the logical principle of contradiction. To establish the truth, one opposes the nonsense to the contradiction, in the same way as in classical logic the negation of negation produces the affirmation, or that in arithmetic minus multiplied by minus is equal to plus. The principle of contradiction was highlighted in the Middle Ages in the Square of Opposition, which I illustrated in Game 4 in the form of a "battle of syllogisms" using playing cards. To conclude on this point, I explicitly introduce in a more general Formula III the nonsense and contradiction principle as follows:

Formula III. The general formula of the "logical nonsense":

Carrollian logical nonsense = "nonsense" + "contradiction"

The logical nonsense remained hidden in Lewis Carroll's work for a long time, until he published the *Game of Logic* (1886) and *Symbolic Logic* (1896). In both books, which present many application exercises for children, the purpose of Lewis Carroll's texts is remarkably explicit[82]. It is not a question of teaching nonsense to children, but providing them an easy and entertaining method that allows them to identify sophisms, solve syllogisms and sorites, thwart false paradoxes and dilemmas. Theses concepts of *reductio ad absurdum* and 'logical nonsense' are illustrated in Games 3 and 7 in particular. In practice, another important point of my research was to see how authors, illustrators and publishers approach the question of the image/text ratio. This led me to carry out the following four case studies.

[82] 'It (Logic) has cost me *years* of hard work: but if it should prove, as I hope it may, to be of *real* service to the young … such a result would more than repay ten times the labour that I have expended on it.' L. C., Christmas, 1896, Preface to the fourth edition of *Symbolic Logic*.

2.6 Four case studies[83]

From about fifty illustrated books, I compare different media: digital, illustrated books and albums for children, pop-ups and games. The aim is to draw a lesson for my illustrations.

2.6.1 Digital Technology: The text/image-sound ratio

The development of the e-book, the audiobook, three-dimensional printing, and the transmission of sound linked to the image have imposed new constraints. For example, there are "best practice recommendations" for image-to-text ratio in emails and emarketing[84]. Some research recommends a ratio of 60% text and 40% images, about 2/3 text and 1/3 images, and not more than three images per transmitted page. More generally on the marketing side, other research proposes the traditional 80/20 of Pareto, 80% of text and 20% of the image. This promotes the text and leaves less space for the illustrator. These are empirical studies and information transmission constraints. For the moment, they do not directly concern the field of illustration of books and paper albums. However, originally, a tale is made to be told orally. Hence the recordings of the tales (audio CDs, MP3 files, etc.) that nowadays accompany the books. This is an example of an extension of the classical image/text ratio. Today, we can speak of a 'text/image and sound' ratio.

2.6.2 Books and illustrated albums for young people

I first analysed the four versions of Alice's adventures: the manuscript (1864), *Alice's Adventures Under Ground,* illustrated by Lewis Carroll, the public edition of *Alice's Adventures in Wonderland* (1865), the shorter version of Alice (1890) for children aged 0 to 5 years, *The Nursery 'Alice'*, and the continuation of the adventures in *Alice Through the Looking-Glass and What Alice Found There* (1871), illustrated by John Tenniel. Next, I made a comparison with three modern illustrators of *Alice's Adventures in Wonderland*, namely: Anthony Browne (2015), Tony Ross (2015) and Tove Jansson (2018). Ultimately, I included for comparisons the initial version of 'Le Petit Prince de Saint-Exupéry' (1999), illustrated by the author and by Joann Sfar (2019 and 2008). The books, authors, illustrators and statistics taken into account are detailed in the Bibliography. In particular, it is interesting to study the image/text ratio depending on whether the publication is the result of a discussion between the author, the illustrator and the publisher, or for posthumous texts between the illustrator and the publisher only (and occasionally with the author's descendants).

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[83] See full case studies and statistical studies in appendix (bibliography II).

[84] Clancy, C., 2019. Return Path. *Best practices for image-to-text ratio in HTML email.* [online]. Available at: <https://help.returnpath.com/hc/en-us/articles/220337107-Best-practices-for-image-to-text-ratio-in-HTML-email> [Accessed 14/6/19]. EmailUplers, 20 March 2017. Medium Corporation. 80:20 – *The New Ideal Text: Image Ratio for your emails.* [online]. Available at: <https://medium.com/@emailmonks/80-20-the-new-ideal-text-image-ratio-for-your-emails-e88d402a7097> [Accessed 14/6/19].

Sometimes even the public finds in a new edition that there has been a posthumous betrayal of the original illustrated version[85]. Added to this are the problems of translating the image/text ratio into several languages. Wordplay and nonsense are not easy to translate and can even lead to different interpretations.
To summarise, when taking the same text, I observe the original illustrations made by the writers (Lewis Carroll and Antoine de Saint-Exupéry) are less numerous than the illustrations made by modern illustrators namely Anthony Browne, Tony Ross and Tove Jansson. Alice's 12 chapters are illustrated with more than 110 illustrations by Tony Ross, in comparison Tenniel made 42 illustrations (in the original manuscript he restricted himself to 37). Joann Sfar's *Little Prince'*s album includes 59 illustrations and his comic strip 660 images, compared to 47 for Saint-Exupéry. Also, the formats of the books (3/2, 5/4, 7/5, 1/1) rarely correspond to the golden ratio (about 8/5). There are exceptions with the *Little Prince* in the Folio collection or with the complete works of Lewis Carroll and the *Little Prince* published in the Pleiade[86] collection where the format 11 cm x 17.5 cm is close to the golden ratio (17.5/11 = 1.59… approximately 8/5).

2.6.3 Pop-ups

Creativity sometimes comes from an association of ideas. If I juxtapose: firstly, the idea of introducing a 'third dimension', called 'reasoning', to the image/text ratio with secondly the idea of Leonardo da Vinci's prototypes, models, automatons and robots, it occured to me to design a 3-dimensional illustration. Which leads to the examination of animated books in three dimensions, with tunnels, windows, volvulus, flaps, wheels, tear-off tongues, pop-outs and pull-downs. It is what nowadays is called a "Pop-up". The animated book dates back to the Middle Ages, where mechanisms show the movement of the stars such as the "volvelles" by Raymond Lulle in 1306 or mechanisms by Petrus Apianus in 1524. With the development of children's books in England at the end of the 18th century, illustrated stories became more and more successful throughout the 19th and 20th centuries. Pop-up is undergoing a revival in the 21st century, among children and adults, with illustrators and papers engineers such as Robert Sabuda, David A. Carter, Marion Bataille, awarded by the Meggendorfer Prize created in 1998. I explore 21 pop-ups of 19 illustrators (listed in the Bibliography[87]).

[85] Parents Momes [online]. Available at: <http://www.momes.net/Apprendre/Heros-et-personnages/Le-Petit-Prince/Les-illustrations-du-Petit-Prince-de-Saint-Exupery-a-Joan-Sfar> [Accessed 12 June 2019].

[86] Created in 1923, La bibliothèque de la Pléiade attaches importance to aesthetics. The cover of the volumes is gilded with fine gold (23 carats). Among the best-selling titles, the *Little Prince of Saint Exupéry* is in first place (2007 statistics).

[87] In this sample, several illustrators are featured in the following YouTube video: 'La face cachée du pop-up' (The hidden side of the pop-up). Pop Up NOW – 6 Dec. 2017. [online].
Available at: <www.youtube.com/watch?v=S3C5BxGIapc > [Accessed 29 June 2019].

TABLE 55

21 POP UPS OF 19 ILLUSTRATORS

Su Blackwell and Corina Fletcher (2015), Mathilde Bourgon (2018), Chloé du Colombier (2019), Bernard Duisit (2018), Guillaume Duprat (2018), Dominique Ehrhard (2018), Fabiano Fiorin (2015), David Hawcock (2019), Paul Hess (2009), Véronique Joffre (2018), Gérard Lo Monaco (2014), Ben Newman (2018), Anne Passchier (2019), Paul Rouillac (2015), collective work, J.K. Rowling (2016), Robert Sabuda (2003 et 2009), collective work (Lo Monaco, Duisit) from A. Saint-Exupéry (2018), Elena Selena (2018), Philippe Ug (2014 et 2018).

It is a fascinating field of experience and creativity, where the three dimensions are part of the story. While respecting a balance between image and text, the pop-up invents a language. By integrating movement, it allows children, as well as adults, to understand or imagine a story without necessarily understanding the language or text. Compared to the book or illustrated album, the number of pages decreases significantly and the number of images increases. In pop-ups that target a young audience, the Pareto ratio is rapidly reached: 80% of animated images and 20% of the text. On the sample examined, there are between twenty and thirty illustrated animations covering ten to twelve pages. There are exceptions, such as the *Little Prince*'s pop-up. The text remains dominant, despite the significant number of animated illustrations (24 in total). In contrast, Robert Sabuda's text is more discreet, hidden behind flaps where the animated image is dominant. However, the sample of pop-ups analysed here do not explicitly introduce reasoning. It is instead in games that I observe the idea of tactics and strategy, which leads me to examine numerous board games.

2.6.4 Games

Lewis Carroll chooses play as a teaching method for children: language acquisition with puns and image games, dialogue awareness and reasoning skills. He invented several games (Word ladder, Word links, and so on), which consists for example of linking two words by a series of similar words. In his manuscript (1864, p. 1), his first drawing emphasises the importance of dialogue and images, 'What is the use of a book, thought Alice, without pictures or conversation?' Children discover chess in *Alice Through the Looking-Glass* (1871): on the 11th move 'Alice takes RQ (Red Queen) and wins'. With the syllogisms of *The Game of Logic* (1886), they discover the basics of logic. Finally, Carroll clearly presents in his work the objectives of *Symbolic Logic* (1896) in the marketing style of today's game publishers:

TABLE 56

LEWIS CARROLL'S GOALS IN LOGIC

'it (Logic) will give you clearness of thought – the ability to see your way through a puzzle – the habit of arranging your ideas in an orderly and approachable form – and, more valuable than all, the power to detect fallacies…' (Lewis Carroll, 1896, Introduction to learners).

Each game develops particular qualities. Some studies[88] have described the benefits for children to play chess from the age of 6 or 7. According to these studies, children who have taken chess classes for two years increase their ability to concentrate by 50%, their memory by 22% and their problem-solving ability by 32% compared to other children. A chess problem is approached as a mathematical problem, with a methodological chain of reasoning. As a mathematician and a logician, teaching children to solve a problem represents a goal for Carroll. He thinks he can reach it through games and entertainment. With this in mind, I analysed about twenty different game categories: card games, draughts, chess and Go games, Monopoly, Scrabble, puzzles, classic game sets, programmable robot buildings, etc. (Bibliography and statistical calculations mentioned in the appendix.)

Two conclusions at this point

Firstly, I deduce three categories of criteria to take into consideration when illustrating games.

The first category concerns **the understanding of the game and its objective**. It is specified in the rules of the game, with an instruction manual, illustrated or not by diagrams.
The second category of criteria concerns **the target audience and communication**. The game editor discloses several details: the recommended age, the number of players, the duration of the game, and so on.
The third category concerns **the technical information**. According to the publishers and the regulations in force in the countries distributing the game, there are several references, mainly (in the sample examined): the size of the box, the dimensions of the game board, foldable or not, the height of the boxes, the weight of the set or box, the number of pieces or objects (counters, figurines, dice, etc.). For example, for chess competitions, there are the following constraints: size N° 5: 32 chess pieces with storage box, apron 45 cm x 45 cm x 1.3 cm, boxes 5 cm x 5 cm, pieces: king height 9 cm (base 3.5 cm). Pawn height 4.5 cm (base 2.5 cm). Regarding the safety standards: cards with rounded ends, no sharp or dangerous objects, etc. The age of use is specified in the game.

[88] BELLAÏCHE, G. Échecs Club de Villeurbanne, 2019. [online]. Available at: http://www.echecsclubvilleurbanne.fr/pages/offre/les-bienfaits-du-jeu-d-echecs.html [Accessed 27 June 2019].

Secondly, this research gave me the idea to use the concept of the pop-up game, which consists of associating the idea of the pop up with that of the game, according to the following formula:

$$\boxed{\text{Pop-up game} = \text{pop up} + \text{game}}$$

The concept of 'game' here aims to associate reasoning with a pop-up. I use it to make concrete the idea of taking reasoning into account in the image/text ratio. Furthermore, the case study highlights the need to identify two kinds of criteria: quantitative and qualitative. The quantitative criteria make it possible to measure several elements, namely, the number of pages, chapters, words in a text, the number of illustrations per book, chapter, idea, book format, game aprons, game boxes, number of players, age of participants, time in minutes of the games, etc. The qualitative criteria specify the objectives: entertaining, facilitating memorisation, concentration, speed, manipulation of objects, reasoning and decision support, etc. These criteria can be grouped according to the objectives pursued and specific constraints. On a practical level, I propose, in conclusion of this first part, to retain nine categories of criteria.

Chapter III

Nine categories of criteria

To illustrate through the visual arts deductive reasoning specific to the pure sciences (mathematics and logic) and, with this objective, to create prototypes in the form of games for children, I have drawn up a checklist of points to take into account. These criteria could be used, completed or modified by other researchers according to their objectives.

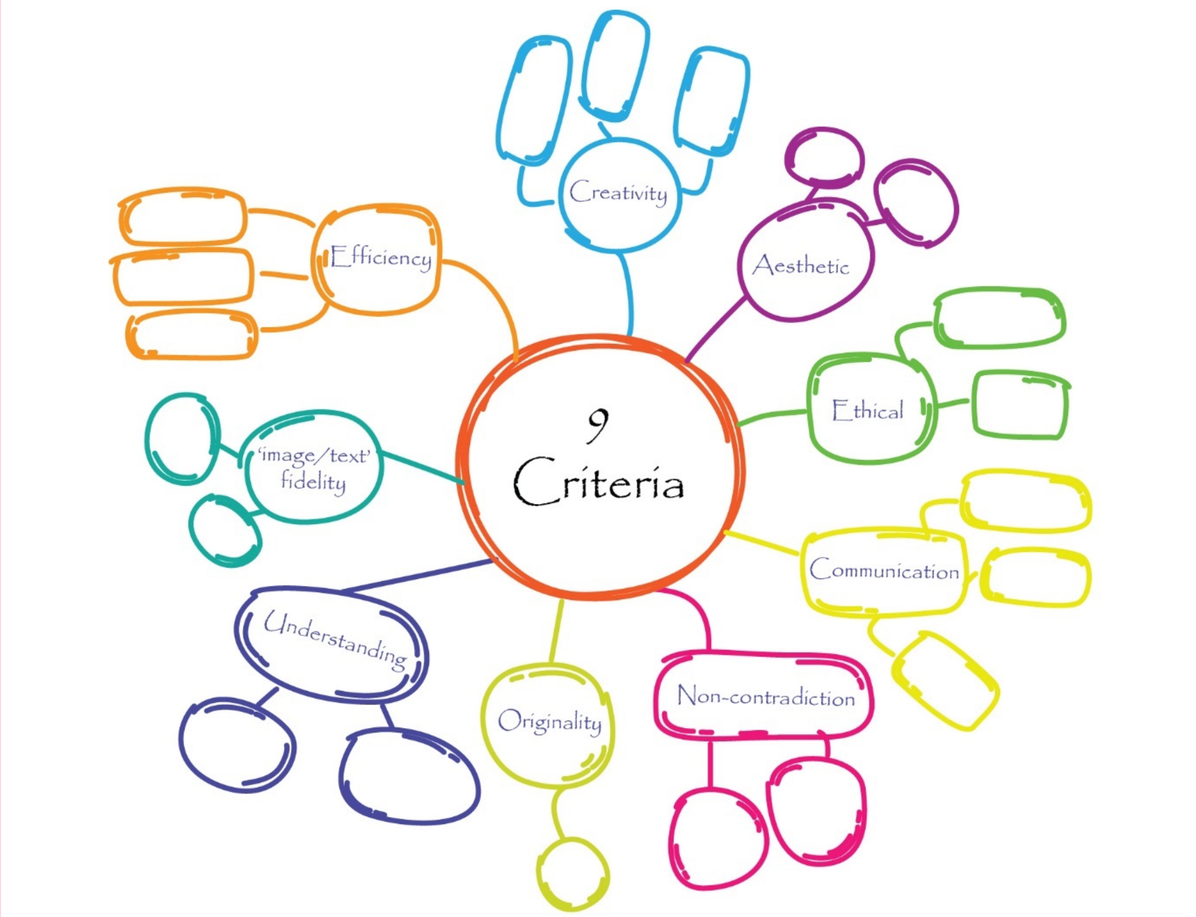
3.1 Method for selecting criteria

Several creative methods can be operated to design selection criteria, as well as classifying and hierarchically structuring them. One example is the heuristic schemes or 'mind-mapping' by Tony Buzan, which is associated with the creative tools of Edward de Bono[89]. In *How to Mind Map*, Tony Buzan (2002), the inventor of *Mind Mapping*, which has since been widely copied, advocates starting from a central circle: the subject studied, summarised in one word or by a drawing.

[89] Among the numerous publications of these two authors, translated into several languages, reference is made here to two books in particular: Buzan T., 2002. *How to Mind Map*. London: Thorsons. Bono E. de, 2007, *How to Have Creative Ideas. 62 exercises to develop the mind*. London: Vermillon.

Starting from this circle, branches and sub-branches are drawn, without any two branches or sub-branches meeting. On each knot is drawn a new circle in which a keyword is written and so on. This is ultimately a visual transposition of Socrates' method, called 'maieutic'. The maieutic method consists, as its name suggests, in giving rise to ideas through a series of questions that permit, if not to find the answer, at least to reflect thereon. It is possible to push the reflection as far as desired, as in the Platonic dialectic.

MIND-MAPPING



The list of criteria that have been drawn up here is based on previous developments and in particular on the results of the case studies outlined above. As with any classification, the choice of terms and their ranking is arbitrary. It has a mainly practical interest. Thus, it is through the practice of drawing that I have decided whether or not they are useful.

3.2 Definition of the categories of criteria

A category of criteria may group together several criteria. Two examples:

1. The term 'communication' is used to classify the quantitative and qualitative 'information' provided by game publishers: age, materials (number of cards, tokens, etc.), the educational benefits of the game, etc. These are the twenty or so items that I take into account in the Booklets associated with my game prototypes. The keyword 'communication' also makes it possible to use criteria resulting from discussions between the author, the illustrator and the publisher concerning the image/text ratio (number of images per page or per book, nature of the illustrations, etc.), as well as marketing criteria for the distribution of the games. One can also hope to be able to envisage a criterion of universality for works that could last over time, such as theorems in mathematics or the essential laws of physics, or even timeless games (Monopoly, the game of the 7 families, chess, etc.) that pass through time.

2. When it comes to illustrating scientific texts, the keyword 'comprehension' or 'understanding' plays an important role. It is necessary to start by understanding the scientific concepts to avoid inconsistencies between the image and the text. Here we find the classical criteria of the image/text ratio set out above, redundancy, conjunctions, disjunction, coherence and non-contradiction.
It should be recalled the objective here is not to illustrate a documentary text or an article of popular science but to use illustration as a visual means of argumentation and problem-solving. The study principally deals with the logic and theory of the syllogism. To do this, I propose to create and illustrate what I call a 'Pop-up game' which I define by the equation:

$$\text{Pop-up game} = \text{pop-up} + \text{illustration of the reasoning used to solve a problem}$$

I, therefore, establish a difference between a classic pop-up and a pop-up game thus defined. In Robert Sabuda's *Alice au pays des merveilles* (2004, Seuil Jeuness), pop-ups, the first page of the paper accordion is unfolded under the title 'Open me', 'Pull me up and look inside', and when one looks through the spyglass, it becomes clear that the pop-up illustrates effectively the title of the chapter and Alice's gradual fall into the Rabbit hole. She has altogether passed through the five floors of this hole filled with bookshelves. However, in a pop-up game (in the sense that I define it[90]), the illustration must not uniquely describe the premises and the conclusion, but equally, make it possible to move from one to the other. In particular, to determine whether the conclusion is true, false or undecidable and whether the reasoning is valid, invalid or undetermined. It is for this purpose that use is made of the circles in the Venn diagrams (Game 5) and the squares in the Lewis Carroll diagrams (Game 6) to show how one arrives at valid conclusions. To do this, two criteria need to be taken into account: the language of the premises (a proposition can be true or false) and the metalanguage (the language of axioms and rules of reasoning that determine whether the conclusion is valid or not). These two criteria are taken into account in the generic category: 'criterion of understanding'.

To sum up, the nine categories of criteria can be represented in a double-entry cross-table, indicating in the first column these nine categories and in as many columns as necessary criteria attached to them.

[90] Some game distributors use the term 'pop-up game', but the term used does not exactly correspond to the definition proposed here.

TABLE 57

NINE CATEGORIES OF CRITERIA FOR ILLUSTRATION

Ranking	List of categorical criteria	Classification by column of related criteria: columns 1, 2, 3, etc. Some examples of criteria.
1	Criterion of understanding	Historical background (Aristotle and the Stoics, Russell, Boole, etc.). Language and metalanguage (premises and rules).
2	'image/text' fidelity criteria	Criteria of redundancy, conjunctions, disjunctions.
3	Efficiency criteria	Criteria of consistency, completeness.
4	Logical criteria of non-contradiction and verifiability	Criteria for no contradiction in the image/text ratio.
5	Criterion of the audience, communication and universality	Criteria for author - illustrator - publisher collaboration, marketing criteria, instruction manuals, material and rules of the game, universally accepted rules.
6	Criterion of creativity, innovation and discovery	Interdisciplinary discoveries and their use. For example, the link between science and art in the works and prototypes of Leonardo da Vinci.
7	Originality criteria	For example, the concept of the pop-up game applied to the field of logic.
8	Aesthetic criteria	Criteria used practicality, 'beauty' of an equation, golden ratio, sequence of numbers of Fibonacci and Beauty of Nature.
9	Ethical criteria	Regulations in force, moral constraints, specific criteria for children's publications, illustrations and games.

3.3 Combination of the categories of criteria

The first four categories of criteria are the most objective. The last six categories of criteria are more subjective or institutional.

The first four categories of criteria form the basis of the thesis here. The aim is to transmit well-established knowledge through the visual arts. Knowledge of mathematics and logic is both universal and timeless. This is the case of *Euclid's Elements* and *Pythagoras' Theorem* or *Aristotle's Organon*. Here, it is not a question of expressing an opinion through drawing, but of making this knowledge accessible and understandable to non-specialists and in the case of this thesis, children in particular. This does not prevent adopting subjective criteria, such as a method of communication (criterion 5) which allows abstract and complex concepts to be visually displayed. With regard to subjective criteria, the artist is free to consider, for example, the golden ratio – which expresses a mathematical ratio of averages – as an aesthetic criterion (criterion 8) and to choose to use it.

Criteria six and seven are common to several disciplines: creativity, innovation and discovery. It includes heuristic diagrams, mind mapping, brainstorming tools, model and prototype creation, experimental methods and inductive reasoning by analogy, metaphor, etc. The criterion of originality highlights the more specific question of the creation of an alternative artistic genre of illustration, such as illustrating abstract concepts and reasoning. With the originality criterion, the illustrator can differentiate himself with a unique style or, on the contrary, by the possibility of adapting to different styles, according to the request of the author, the publisher and the target audience: stories, novels, fictions, documentaries, and scientific works. The eighth criterion is the aesthetic criterion, as already proposed by the Roman architect Vitruvius in the 1^{st} century BC. A pop-up is a three-dimensional architectural construct to which it is possible to apply the three *vitruvian* criteria: *firmitas* (solidity), *utilitas* (utility), *venustas* (beauty). The ninth criterion is that of ethics. It refers to intellectual honesty. This becomes an increasing problem with the popularisation of scientific works. This criterion remains on top of a moral issue. Not all drawings are suitable for children.

To conclude on chapter II

This representation of the nine categories of criteria in the form of a cross table only takes into account 2 dimensions, that of rows and columns. It seemed necessary for me to be able to combine the criteria, depending on the objectives and priorities that are set. For example, in the context of children's books for educational purposes (acquisition of knowledge, development of the capacity for deduction and logical reasoning), it is useful to be able to combine purely educational criteria with criteria of games and entertainment and to apply ethical and aesthetic rules. To combine the criteria, I was inspired by two tools: the slide rule and the colour circle represented on superimposed rotating discs (chapter VI).

By turning the wheels, one can form triads and examine different colour combinations. This concept is particularly useful here to be able to combine the comprehension or understanding criteria with the other categories of criteria. Moreover, as will be seen in the next chapter, this makes it possible to distinguish visually between language and metalanguage and particularly within metalanguage the rules of logic and the rules of pop-up games as explained in the second part that follows. In total, these nine categories of criteria remain a practical way to design and evaluate the relationship of images to text in the 'three dimensions' mentioned. According to the principle of combinational art, they can be combined two by two, three by three, and so on.

Part II

Visual Arts and Art of thinking: From practice to theory and from theory to practice

This second part highlights the value of using the metalanguage of the pure sciences and its rules to illustrate abstract concepts and complex reasoning (Chapter IV). The term 'metalanguage' is used in this thesis in order to distinguish, on the one hand, the common practice of a language, and, on the other hand, the axioms, rules and signs that make up the 'grammar' of a language. The illustration of the Art of Thinking, i.e. the logic and its historical evolution is done through its main concepts, axioms and rules. I then discuss the choice of a model to illustrate this metalanguage (chapter V). Finally, I show how to move from these theoretical points of view to the practice of illustrating through games the main concepts of logic using visual arts (chapter VI).

Chapter IV

Language and metalanguage

4.1 Distinguishing between language and metalanguage

Some definitions[91].

1. In current language, the words, vocabulary and rules of grammar in an English text (read, written, spoken) are formulated in English, that is, in the same language. Let us call 'L' this language that allows us to express ourselves: L = the English language.

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[91] Gensler, H. J., 2002. *Introduction to logic*. Reprint 2017, 3rd ed. London, New York: Routledge, Taylor & Francis Group. The author gives an introduction to 'metalogic', that is, the study of the concepts and rules of logic (Chapter 15, p. 334). Peeters, M. and S. Richard, 2009. *Logique formelle*. Wavre, Belgique : Éditions Mardaga, Cosmo-Logiques Collection, pp. 25–30.

2. To judge whether the spelling of the words is correct, if the grammar rules are correctly applied, it is necessary to employ a language that speaks the same language as the language. It is a metalanguage (L +1), from the Greek 'meta', above or next to the language (L). This metalanguage (L + 1) will be expressed in English. In order not to confound it with common language (L), it must be distinguishable. For example, in writing, the common language 'L' will be employed in quotation marks. Orally, it is specified by a gesture simulating the quotation marks with our fingers.
3. This metalanguage (L + 1) has a practical use. It serves to convey a grammatical, logical, aesthetic judgement on a sentence (L) of the current language.

TABLE 58

LANGUAGE AND METALANGUAGE: AN EXAMPLE.

If one says: 'This sentence has five words,' it is true.

The first part: 'This sentence has five words,' , is written in the current language L, while the judgement 'it is true', although written in current English (L), is part of the metalanguage (L +1). It is a judgement on what is stated. The confusion of the two languages L and L +1 creates sometimes amusing misunderstandings. It is often at the root of the paradoxes from which it is sometimes difficult to escape.

For example, when Magritte writes over a perfectly well-drawn apple, 'This is not an apple' or under the design of a pipe, 'This is not a pipe,' there is a visual overlap of two languages. This cannot leave anyone indifferent. Here, the image/text ratio is paradoxical. From the point of view of the image, Magritte's apple and pipe represent an apple and a pipe. Due to the fact every image is a fiction, the image tells the truth. However, from the point of view of the text, no one can bite this apple or smoke Magritte's pipe. So, Magritte is right to say: 'This is not an apple' and 'This is not a pipe'. This contradiction between image and text expresses a paradox.

The distinction between current language and metalanguage is particularly important in logic and reasoning. The metalanguage makes it possible to understand what formal sciences and natural sciences (based on reality and experience) have in common and what distinguishes them, as defined in the following table.

TABLE 59

SYNTAX AND SEMANTICS IN THE METALANGUAGE OF PURE AND NATURAL SCIENCES

– The so-called syntactic metalanguage deals with axioms, postulates, theorems, demonstrations, rules of inference (deduction, induction, analogy). As in common grammar, the rules are part of the syntax. There are rules of syntax in the pure sciences as well as in the natural sciences. For example, to know if a proposition is a theorem, it is necessary to analyse the rules of successive transformations which have made it possible to move from axioms to a conclusion in logic or to move from one theorem to another in mathematics.

– The so-called semantic metalanguage deals with relationships that link sentences or expressions in current language L and specify their meaning. For example, saying a proposition is true or false, a conclusion is valid or invalid, is part of semantic metalanguage. There are rules of semantic in the pure sciences as well as in the natural sciences.

These two syntactic and semantic metalanguages are complementary, the first is considered by Willard Van Orman Quine (1986) as the 'science of deduction', the second as 'the science of truth'. If pure sciences and natural sciences have their language and metalanguage, why shouldn't illustration and visual arts have their ones? This is the question I asked myself. The current language (L) of illustration and visual art is what we see, feel, understand (or do not understand in the abstract art, for example). When Daniel Arasse[92] talks about abstract art and writes, 'we do not see anything', it is indeed a value judgement that he makes about this art. This judgement belongs to the metalanguage (L +1). If illustration and visual arts have their metalanguage, it means they have their axioms, rules and criteria. For example, one can consider the golden ratio is for some artists a criterion of Beauty, without the need for them to justify themselves. Because an axiom is by definition neither true nor false. It is indemonstrable and must be accepted as such. This does not prevent changing it if it is not satisfactory. Art also has its own rules, such as perspective and trompe l'oeil, for example. This amounts in practice to defining axioms and rules in illustration and visual art as they exist in the pure and natural sciences.

[92] Arasse, D., 2000. *On n'y voit rien*. Paris: Folio Essays, Denoël éditions. It is a deliberately provocative text by this specialist and author of a thesis at the Sorbonne with André Chastel on *Italian art and the Renaissance*.

What is even more interesting to note here is that while all disciplines are distinguished by their language, they often have metalanguage in common. This is particularly true for formal logic and pure mathematics. At least this is what I have observed. According to my criterion N° 1 (first to comprehend what it is, before illustrating it), I had to reread more carefully books on logic under this aspect of metalanguage. I took notes on this occasion in the form of cards (140 cards, size 10 cm x 15 cm, written on both sides). A problem instantly arose. How to classify them? To classify the cards, I surrounded them with a coloured line, red for the definitions and the concepts, yellow for the rules of deduction and validation, blue for the results. This classification broadly corresponds to the grouping of the main parts of Tricot work on logic[93]. This classification employing colours (which I will take up again later) allowed me to observe that the three major arts of logic are at the very heart of mathematics.

TABLE 60

THE THREE MAJOR ARTS OF LOGIC

1. The art of definition, classification and concept.
2. The art of judgement (truth - false, just - unjust, beautiful - ugly, good - evil, etc.).
3. The art of demonstrations, proof and calculation.

The term ‘art’ is used here, in the sense of know-how or practice. It is this metalinguistic art which is common to several scientific disciplines. The metalanguage of logic is interesting to illustrate since it makes it possible to move from one discipline to another. Besides, it is the basis for understanding the foundations of a theory through its concepts (axioms, definitions and classification), rules (reasoning, validation and judgements) and concrete results. The main metalinguistic principles of logic can be found in almost all classical scientific disciplines.

[93] I owe the idea for this presentation of logic to Jules Tricot. He translated the logical works of Aristotle known as *Organon,* published by the Librairie Philosophhique J. Vrin, Paris (Vol. 1 to Vol. VI, reprint 1995–2014). Tricot, J., 1928. *Traité de logique formelle*. Reprint 1973, 3rd éd., Paris: J. Vrin.

TABLE 61

THREE MAIN PRINCIPLES OF LOGIC

1. The principle of identity which affirms that A is A and nothing else.

2. The excluded third party principle (or excluded middle) affirms that a thing (or a proposal p) is or is not; therefore the disjunction p OR non-p is always true.

3. The principle of non-contradiction states that the same property cannot at the same time both belong and not belong to the same object in the same respect. Consequently, the p AND non-p proposals cannot be simultaneously true. In other words, the conjunction p AND non-p is always false. Thus, two statements are in contradiction when what one says denies what the other says.

These principles can be found in the application of the image/text ratio in the form of an identity between image and text and the avoidance of contradiction between the two languages, as in the cited example of the Apple or Magritte's Pipe. In short, I concluded that if I wanted to illustrate classical logic, it was these three fundamental principles that I had to illustrate first and foremost. It is undoubtedly what Lewis Carroll thought when he imagined the *Game of Logic* and wrote *Symbolic Logic*, and taking inspiration from Venn's diagrams created his own diagrams.

4.2 Distinguishing axioms and rules of inference and validation in metalanguage

In its metalanguage, logic distinguishes between diverse types of concepts, reasoning rules and validity criteria, such as the concept of the truth of a proposition or the validity of reasoning. Consequently, the illustration of these concepts needs to differentiate them, namely:
—the primary concepts and objectives (noted 1a) and
—the metalogical rules, whether they are a reasoning rule (noted 2a), or a validation rule for inference (2b.1) or the validation of results (2b.2) according to the objectives pursued.
In other words, to illustrate a theory as the one of logic through the abstract concepts of its metalanguage, this requires illustrating the three main points of a deductive reasoning shown in the following table.

TABLE 62

STRUCTURING A DEDUCTIVE REASONING

In three points:

1. Primary concepts and objectives (noted 1a).

2. The rules, which are of two kinds: rules of reasoning (noted 2a) and rules of verification or judgement (noted 2b), which are themselves of two kinds and can serve as criteria:
(noted 2b.1) to validate the reasoning rules – these are rules of logical coherence, simplicity, elegance in formulation, ease of communication – and rules (noted 2b.2) to validate the results obtained according to the objectives initially set.

3. The results (expected or unexpected) according to the objectives and primary concepts (or assumptions 1a) initially set (noted 3).

Concerning the reasoning and validation of logical rules, as indicated by Gijsbers (2017) among others[94], it is essential to distinguish between purely deductive sciences and experimental inductive sciences.

TABLE 63

DISTINGUISHING BETWEEN PURE SCIENCES AND EXPERIMENTAL SCIENCES

In the pure sciences (mathematics and logic), the primary concepts asked to be accepted at the beginning of a demonstration are axioms and postulates (sometimes called hypotheses). They are not demonstrable and not refuted. The rules of inference (or reasoning) remain rules of deduction. Deductive thought naturally progressives from the general case to the particular and not the reverse.

In the empirical and experimental sciences, the first concepts are always refutable hypotheses. They need to be verified. The rules of inference or reasoning are initially inductive. Their generalisation then requires the use of deductive reasoning. Thinking starts from the observation of particular cases to try to deduce from them general laws, if possible universal.

To establish the difference between the first concepts (axioms, postulates) and the objectives and rules of inference (reasoning by deduction, induction or analogy), it requires a deeper understanding of the notions of concepts and rules, notably, if the goal is to be able to illustrate them. I am referring here to Claude Panaccio (2001) and Jean-Pierre Cometti (2011) to clarify the question of concepts and rules.
Firstly, what is a concept? Panaccio (2011) considers that a concept must meet 5 conditions. I summarise (and translate into English).

[94] Dr Victor Gijsbers, *Introduction to Logic*, Leiden University – Faculty of Humanities, 14 Sept. 2017, ch. 1.1 [online]. Available at: < https://www.youtube.com/watch?v=K4ChzesrWKI> [Accessed 16 June 2020]. And, Poincaré, 1902 ; Popper, 1963; Thibaudeau, 2006.

TABLE 64

FIVE CHARACTERISTICS OF A CONCEPT

1. be conceivable by all of us,
2. have a possible extension,
3. be able to combine with other criteria,
4. be learned, because it is not innate,
5. be shared by several people.

From this point of view, it is possible to admit that metalanguage is a concept that meets the five criteria mentioned. Since the objective of my research is to make children understand abstract and complex reasoning through visual arts, if one admits that metalanguage is an abstract and complex concept, the consequence is that it must be illustrated. In the field of art, as seen above, the golden ratio taken as an aesthetic axiom can be illustrated by Fibonacci's sequence. Another example is the rules of perspective. They can be classified among the rules of metalanguage. The question here is to illustrate the axioms and rules that constitute the metalanguage of logic, and much of pure mathematics. This is the subject of the Games 1 to 7 and their associated Booklets.

Second question: what is a rule? According to Jean-Pierre Cometti (2011) a rule can be defined by its characteristics. I summarise (and translate into English):

TABLE 65

FIVE CHARACTERISTICS OF A RULE

1. A rule is a principle of action, thoughtful, ordered, methodical, subtracted from chance.
2. It indicates what needs to be done to achieve this or that goal.
3. It needs to be implemented.
4. Its criterion is efficiency and its risk is not to be applied correctly.
5. It must make it possible to say why it is correct and why it is not correct.

With these characteristics, the nine categories of criteria outlined above can be seen as rules for illustrating abstract concepts and reasoning. They can be employed to differentiate between a primary concept (called axioms) and a rule. They indicate what needs to be done. For example, category 4 will be used to check there are no contradictory propositions in given reasoning. This category can be used as a criterion for validating the results obtained under the objectives set. These nine categories of criteria and their possible combination are not purely arbitrary.

Drawn from case studies (part I, 2.6 above) and my own experience, I have used them to make the pop-up games and Booklets. Other researchers will be able to take them up, modify them or complete them according to their subject of study. The principle to be retained here is the fact of establishing rules, which are constraints that I have tried to respect as much as possible. This has been useful to me in practice.

4.3 Choosing a theory to illustrate abstract concepts and complex reasoning

I resume. From the 140 cards of a 10 cm x 15 cm format that I compiled in the manner of Le Sage (1724–1803, cited below), I summarise here the lessons that I deduce in three points. Firstly, I observe that a theory consists of two languages: the current language with its signs and symbols, and the metalanguage with its axioms and rules that specify how to use the current language.

Secondly, this approach makes it possible to define and classify three categories of concepts: 1) primary concepts called axioms, hypotheses or principles, 2) concepts of validation or judgement, 3) concepts of demonstration and proof which, based on the reasoning and judgement, produce expected or unexpected results.
Thirdly, by definition, the concepts of a theory have several possible extensions. They can be combined with the concepts of other theories, and be shared, learned and used by many people in different fields of human activity. One word sums up this principle, namely: ‘interdisciplinarity’ or the sharing of knowledge in different fields of activity. For example, the concepts of the golden ratio, the Fibonacci sequence of numbers and the geometrical laws of perspective allowed Leonardo da Vinci to establish a bridge between art and science. Another example of multidisciplinarity is the principle of contradiction. It is undoubtedly the most important axiom of classical logic. It is an abstract principle, more metaphysical than physical (Aristotle, *Metaphysics*, Gamma Book, chap. 3, 1005 b 19–20). By definition, it is undemonstrable and should, therefore, be classified among the primary axioms and concepts. It can equally be employed as a criterion for judgement and reasoning.

Conclusion

The conclusion I draw from these three main points is that it may be useful to start by identifying concepts that can be used to bridge the gap between visual arts and pure sciences. The principle of non-contradiction is an example of a criterion to be retained. It makes it possible to judge in the sciences as well as in current discourse the falsity of two contradictory propositions. Moreover, it serves as reasoning in what logic calls *reductio ad absurdum*. As seen above, it acts as an antidote to nonsense in paradoxes. In the field of illustration, it can be used, for example, to identify a contradiction in the image/text ratio or to highlight nonsense in a cartoon.

Since the laws of thought have developed over the centuries, there are several models of theory and several types of reasoning. Hence the question: Which model should be adopted, adapted or invented to illustrate the metalanguage of pure sciences, and in particular that of logic? It is the subject of the following chapter.

Chapter V

Discussion on the choice of a model to illustrate the metalanguage of pure science and logic in particular

A first methodological point to be clarified is the relationship that can be established between theory, research and the practice of sciences and arts.

5.1 A methodological aspect concerning 'Practice-Based Research' and 'Practice-Led Research'

Concerning the link between theory and practice, the question has been raised for illustration purposes. It has led to the distinction between 'Practice-Based Research' and 'Practice-Led Research'. To complicate matters, several authors have observed the terms 'Practice-Based' and 'Practice-Led' are frequently used interchangeably. In her Guide, Linda Candy (2006) and in another similar way Ayer (1956) define these concepts as follows:
— If a creative artefact constitutes the basis of the contribution to knowledge, the research is 'Practice-Based'.
— If the research leads primarily to new understandings about practice, it is 'Practice-Led'.

Without having to deepen this debate here, I consider that these two approaches can be complementary. Without employing these definitions, this was assuredly Leonardo da Vinci's thinking, as we saw in the first part. For him, practice supports theory and theory supports the practice. At least that is the point of view I have taken here throughout my research, both practical and theoretical. For this, I refer in particular to Graeme Sullivan's *Art Practice as Research: Inquiry in Visual Arts* (2004)[95].

> 'In considering the practice of research it is necessary to distinguish between method and methodology,' he writes (Sullivan, 2nd ed. 2010, p. 35).

It seemed particularly legitimate to me to distinguish in my research between method and methodology. I give the term 'methodology' a broader meaning than 'method'. Because in the same way that a concept is characterised by the possibility of being used by a large number of people, my idea is that the nine categories of criteria method that I have initially designed should not constitute a single method that can be employed.

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[95] Graeme Sullivan is Professor of Art Education, Teachers College, Columbia University. Since the early 1990s, he has been researching the critical-reflective thinking processes of artists and methods of inquiry used in visual arts.

In my view, the design of each category of criteria should be established according to the subject matter, based on the study of different scientific and artistic methods. The aim is to establish interdisciplinarity between researchers using the visual arts to work on abstract concepts. On the methodological level, the concept of 'theory' and 'empirical research' have rarely been combined in the history of philosophy, until scientists discovered, particularly in theoretical physics, that the two approaches are complementary (Dekens, O., 2005). The formal language of mathematics has become a common language in science, including in the empirical sciences. Myself, to illustrate the theory of the syllogism, I had to associate the practice of drawing with the theory of logic. In terms of the use of visual arts, the problems became more difficult to solve as the concepts to be illustrated from logic became more abstract. For creating games and pop-ups, especially Games 1 to 7, theory and practice had to be linked here.

To sum up, the challenge here was to develop, not a method, but a methodology taking into account several theoretical and practical methods of science and art. It will be an opportunity to verify the following proposal of the American psychologist Kurt Lewin (1890–1947): 'Nothing is more practical than a good theory.'[96] Furthermore, I have found that what makes it easier to switch from one method to another, and more generally from one language to another, artistic or scientific, is the metalinguistic concepts they have in common, such as the use of axioms and rules. There remains a crucial question. What theoretical or experimental methodology should be adopted as a priority for illustrating abstract concepts: the deductive model of the pure sciences, the inductive model of the experimental sciences, or the models based on analogies? Hence the following discussion.

5.2 Which model should be chosen to illustrate abstract concepts?

As Graeme Sullivan explains (ed. 2010, p. 67), theorising involves adopting a problem-solving strategy that has already been proven in various areas of knowledge. It is what he designates in practice, 'Using problem-solving strategies' and 'also making use of information from other fields if it helps to achieve a successful solution'. This approach assumes the art researcher is interested in several disciplines: philosophical, theological, scientific fields, etc., the knowledge being multidisciplinary. Here, Graeme Sullivan presents several lines of thought that can be gleaned from his important bibliography. The question submitted is: How to visualise abstract ideas and concepts? Among the models that Graeme Sullivan evokes is the model of *analogical reasoning* applied to illustration that seems closest to art: visual analogy, visual metaphor, visual homology (ed. 2010, p. 196). However, he is himself not completely convinced by the effectiveness of these visual stylistic figures.

[96] Kurt Lewin is known to be one of the first to consider psychology as a 'hard science'. In particular, he is responsible for the concept of 'group dynamics', a major concept of 'industrial psychology'.

TABLE 66

SULLIVAN'S QUESTIONS ON ANALOGICAL REASONING MODELS

– 'Does the visual analogy help translate information in a way that increases understanding?'

– With visual metaphor, 'Does transferring information between two images serve as a useful bridge upon which further conceptual structures can be built?

– Do visualisations, drawn from different classes and genres but are based on similar structural principles, identify a conceptually plausible and generative equivalence?'

Sullivan, G., 2004. *Art Practice as Research: Inquiry in Visual Arts.* Reprint 2010, 2nd ed. Pennsylvania State University, USA: Sage Publications, p. 196.

In addition to *reasoning by absurdity* and *by recurrence*, philosophy of science has highlighted three principal types of reasoning: by induction, deduction and by analogy.

5.2.1 Analogy: A mainly rhetorical figure

It requires *four terms* to establish reasoning by analogy, namely, two subjects and two predicates. It is only a similarity relationship: A is to B, what C is to D. This implies a notable similarity between B and D. That is four terms to take into account in total instead of three in the deductive syllogism. Analogy remains a figure of rhetoric which does not have the same objective as logic. This is what Aristotle had already indicated when he was opposed to the Sophists. Rhetoric seeks to convince by all means (including falsity), while logic seeks truth. One may indeed wonder if the analogy does not lead to an incorrect perception of things, because it is an approximation or similarity with reality. The aim here is to illustrate abstract concepts that are not directly perceptible by the senses, and that generally do not rely on reality. Employing an analogy, such as a metaphor, in metalanguage can cause confusion between what is real and what is not, and ultimately produce a distorted image of the abstract concept. In any case, in practice, analogy seems to be used more often in literature and poetry than in the pure sciences[97]. If one sticks to Graeme Sullivan's idea of choosing models which have proven their worth, there remains the choice between two scientific models. The experimental sciences and the pure sciences (mathematics and logic).

[97] 'Analogy is an unstable means of argumentation,' write Chaïm Perelman and Lucie Olbrechts-Tyteca, in Perelman, C. and L. Olbrechts-Tyteca, 1958. *Traité de l'argumentation*. Bruxelles: Éditions de l'Université de Bruxelles, 6th ed. 2008, pp. 527 and 535.

5.2.2 Comparing inductive and deductive models

The natural sciences principally use the method of induction and experimentation. As Victor Thibaudeau (2006, p. 773) expresses it (I translate into English): 'The movement of intelligence is from the bottom up, by releasing something general from more specific knowledge.' In that manner, physics, chemistry, life sciences are experimental sciences based on the in-depth observation of facts (Grégoire, 1953, pp. 143-158). It is from the observation of reproducible phenomena that general laws are deduced and then verified by experience. This experience-based approach is widely adopted by statisticians. They have highlighted some statistical laws that have become famous, such as the law of large numbers (Gauss curve also called 'bell curve' because of its shape), the law of small samples (Poisson's law), etc.

According to another pattern of thought, pure science uses deductive reasoning that starts from axioms and moves to conclusion**s.** What Thibaudeau considers (2006, p. 719) 'a top-down movement'. The syllogism is the best-known form of deductive reasoning in classical Aristotelian logic. The reasoning starts from the assertion of two categorical premises to arrive at the conclusion. This is the form of reasoning used in Aristotelian categorical syllogism (Plantin, 2016. pp. 182–184). This form of reasoning was completed in the Middle Ages by the Logical Square (or Square of Opposition, illustrated in Game 4).

Contrary to Aristotle's categorical syllogisms, 'hypothetical-deductive reasoning' starts from a hypothesis whose consequences it explores. These are the models found, for example, in the *modus ponens* and the *modus tollens* (illustrated in The Logical Spring and in Game 7). This form of reasoning was improved in modern mathematical logic with Boole's logic (illustrated in Game 7).

The difference between the two modes of reasoning by induction or deduction raises the question of the validity of inductive and deductive theories. Logic teaches that it is not possible to deduce a general law in a particular case, which raises what Karl Popper calls 'the problem of induction' (Popper, 1963, ch.1). It is precisely the role of the validation rules and criteria which allow a judgement both on the result and on the reasoning used to derive a valid conclusion. From this point of view, the following table summarises the distinction that must be established between experimental sciences and pure sciences (Poincaré, 1902).

TABLE 67

THE DIFFERENCE BETWEEN EXPERIMENTAL SCIENCES AND PURE SCIENCES

In experimental sciences, the validity of the result of a theory is determined by the observation of the facts. The veracity of the theory depends on the veracity of its assumptions by comparing them to what can be observed in nature. To do this, more or less sophisticated instruments are utilised: microscope, astronomical telescope, wave emission, particle accelerator, etc.

In pure sciences, logic and mathematics, which have somehow got rid of reality, can no longer resort to the observation of facts. For example, one metre cannot be adopted to verify the distance between two points in a 4-dimensional, 5-dimensional, 6-dimensional or more space. These dimensions are not accessible to us materially, but only by reasoning. Consequently, the veracity of conclusions in logic and theorems in mathematics can solely depend on the validity of the reasoning used, that is, the axioms and rules of deduction themselves, which enable mathematical and logical demonstrations to be carried out.

It is not the role of the illustrator to validate scientific theories. On the other hand, it is important to comprehend how a scientific theory is constructed if the objective is to illustrate it. Among the nine categories of criteria selected, I considered that the criterion of ‘understanding’ was the most important for illustrating scientific theories. This led me to write instruction manuals to acquire the basics of logic. It is these manuals that I have illustrated which will be used by the players to deepen, through Games 1 to 7, the main concepts and modes of reasoning of logic.

Two issues remain to be decided. What to adopt: inductive or deductive reasoning? What is the practical consequence of this choice for the illustration? To make a choice, I refer to Thibaudeau (2006) and Rabau and Pennanech (2016), then I draw the consequences of the choice made.

5.2.3 Choosing the deductive reasoning model to illustrate abstract concepts

Thibaudeau (2006, p. 792) summarises in a table the advantages and disadvantages of three modes of reasoning: deductive, inductive, analogical, according to three criteria which I sum up here.

TABLE 68

CRITERIA FOR CLASSIFYING THE ADVANTAGES
AND DISADVANTAGES OF THREE CLASSICAL MODES OF REASONING
BY INDUCTION, DEDUCTION AND ANALOGY.

First criteria: the way of thought
a) By deduction, thinking goes from the known to the unknown and from the general to the specific.
b) By induction, thinking goes from the particular to the general.
c) By analogy, thinking goes from the similar to the similar.

Second criteria: the number of terms involved in reasoning

a) For a categorical deductive syllogism, three terms (subject, verb, predicate), two premises and a total of three propositions are enough to obtain a certain conclusion.

b) For inductive reasoning an extensive enumeration of special cases and tests are necessary to arrive at a plausible conclusion.

c) For reasoning by analogy, 4 terms including 2 similar are necessary to conclude by similitude.

Third criteria: the qualities and defects

a) The deductive reasoning used in logic or pure mathematics is often abstract and complex. It is the least accessible of the three classical modes of reasoning, even if it is universally the most satisfactory. Once demonstrated, a theorem remains universally true, such as the Pythagorean theorem.

b) The Inductive reasoning is based on examples and practical experience. It may seem easier to access. However, to draw conclusions from experiences, it is often necessary to use deductive reasoning. This then does not simplify its application.

c) Reasoning by analogy is effective, especially in rhetoric, in speeches, or to explain a subject and the definition of a concept that is difficult to understand. However, as Thibaudeau points out, it is recognised that this mode of reasoning is superficial.

Because my research concerns the pure sciences in general and logic in particular, the choice of the deductive model for illustration seemed preferable to me, in particular, to apply n°1 criterion of understanding. This choice should also be able to satisfy the other categories of criteria selected, in particular the creativity criteria. Rabau and Pennanech (2016) show that deductive reasoning can be creative. They propose several exercises (cross-tables, the Logical Square, etc.) that allow deductive reasoning to be used to bridge the gap between literature and science.

The discussion of whether logic and mathematics are creative models as art can be, is far from over. In logic, for example, one might think that deductive reasoning would not be creative because it does not invent anything: the conclusion is entirely contained in the premises. This is true. It is even the definition of the syllogism. While it is also true that the axioms and rules of deductive models are arbitrary concepts, often impossible to prove, there are nevertheless at least six arguments that led me to adopt the deductive model.

Argument 1. It is an advantage rather than a disadvantage that the conclusion of a syllogism is entirely present in the premises. Otherwise, we would be talking about things that have not been defined. The conclusion teaches or confirms things that we would never have thought of naturally. The cascading deduction, from one theorem to another, helps to discover new ideas, new concepts. It is a creative method.

Argument 2. There are many examples of creativity in logic and mathematics. These include classical and mathematical logic, non-standard logic, and the incredible development of theories in mathematics. For example: the theory of symmetry and regularities, set theory, string theory used in theoretical physics, graph theory, game theory, etc.

Argument 3. Pure logic and mathematics represent abstract and formal sciences. Paradoxically, by getting rid of reality, they have become stranger and more creative than fiction. It conducts me to Lewis Carroll's *Humpty Dumpty theory* in Alice in Wonderland, 'When I use a word … it means what I want it to mean, no more and no less', writes Lewis Carroll.

Argument 4. Naturally, one might think that axioms, postulates, rules are arbitrary conventions. Although not refutable, they cannot be totally arbitrary. As Panaccio (2011, pp. 36–54) makes it extremely clear, this is a misunderstanding of this concept. An axiom, a postulate and rules of inference are metalinguistic concepts. They have a role to play, a function to perform, a task to accomplish. In brief, a purpose. Defectively designed, poorly imagined, they will receive little chance of producing interesting results in relation to objectives set. In other words, when creating an axiom or defining a rule, imagination, creativity and intuition are needed to achieve a given objective.

Argument 5. Therefore, axioms in logic and mathematics, as hypotheses in the experimental sciences, cannot be totally arbitrary. As explained by the mathematician, physicist, engineer and philosopher Henri Poincaré (1902) science is based on the principle of finding good hypotheses, which remain a creative problem. As any concept, an assumption is neither true nor false. It is simply employed to develop the reasoning of the logical form, 'either hypothesis A is true, then we can deduce that…'. We find here the *modus ponens* of logic. If p then q or p is true, therefore q is true. On the other hand, the deduction 'If p then q, or not q, then not p' is an incorrect reasoning. This shows the interest of having deduction rules to avoid fallacious reasoning[98].

[98] Examples of correct reasoning: if it rains (p) then I take my umbrella (q). It is raining (p). Therefore, I take my umbrella (q). An example of an incorrect reasoning is: if it is raining (p) then I take my umbrella (q). I do not take my umbrella (non-q). Therefore, it does not rain (not p). In this second case, the reasoning is wrong, because even if it does not rain, I could take my umbrella to protect me from the sun.

Argument 6. Deductive reasoning does not preclude intuition. Mathematical and logical mind often proceeds by intuition and conjecture. A conjecture is a reasonable proposal, but one that has not yet been convincingly demonstrated such as Goldbach's conjecture. Another example is the strength of the demonstration in geometry. It is possible to reason correctly even on a wrongly drawn figure, because the figure is only there to facilitate the demonstration.

In Summary, whether one chooses the deductive or the inductive model reasoning, the essential question is to determine what are the criteria for the validity or refutability of a theory. Given the above arguments and the fact that pure science relies mainly on deductive reasoning, I have chosen the deductive model to illustrate abstract concepts. Furthermore, it is also the model used by Euclid, Aristotle and Lewis Carroll to solve syllogisms. An alternative that other researchers can adopt would be to use the inductive model of empirical and experimental sciences. However, this type of model requires a considerable number of experiments and tests to be carried out, without any guarantee that a generality can be drawn from particular cases (Poincaré, 1902). It remains to be seen what the consequences of this choice will be for the illustration of abstract concepts.

5.2.4 Consequences of the choice of the deductive model for illustration

As in mathematics and logic, I begin with an axiom. I postulate that visual arts are a good medium between art and science. By definition, a postulate is a statement asked to be accepted at the beginning of a demonstration (from Latin *postulata*). Axioms and postulates are unproven principles. They are by definition non-refutable. It prevents the infinite regressions of the genre that define: a by b, b by c, c by d, and so on. While they are irrefutable, it is always possible to alter them if they prove to be of no use in demonstrating the proposals or results sought. In this manner, if it turns out that it is impossible to illustrate abstract concepts through visual arts, it would be necessary and without hesitation to renounce the postulate stating. On the other hand, and this is the difference between a postulate and a hypothesis, a unique counter-example (or several) cannot challenge a postulate[99]. In precis, a postulate as an abstract concept is neither true nor false. It gives more or less interesting results which means even if I cannot illustrate abstract concepts, it doesn't mean other illustrators cannot do it. Another consequence of the deductive model is that it leads to universal and timeless results. The aim is to highlight a way of using the visual arts that can be used to illustrate abstract concepts and complex reasoning in the field of logic, as well as in other disciplines.

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[99] In the experimental sciences that attempt to establish general and universal laws of many experiments, it is enough to find a single counter-example to refute the hypothesis, and thus the whole theory, or almost. At least, this is the conception of the philosopher of science Karl Popper, shared by the scientific community. For Popper, an experimental science is only a science if it is refutable. It is called the Karl Popper's principle of refutability (Popper, 1963 and 1973).

Conclusion

From the previous analysis and the 140 cards I have compiled on logic, I can deduce the following conclusions. To distinguish language from metalanguage is to distinguish between the form of language and its content. In logic, as in mathematics, the representation of metalanguage is most often in the form of symbols, while the language employed is that of the current language specific to each country. A notable example of symbols used in logic is the ideography[100] of Friedrich Ludwig Gottlob Frege (1848–1925). Nonetheless, his symbolic writing was so complex to draw and memorise that logicians and mathematicians did not retain it. Similarly, modern civilisations have not adopted Egyptian hieroglyphic writing, although it is figurative, nor retained to count the additive numbering used by the ancient Romans. This leads me to draw two further conclusions.

Firstly, to adopt the conceptual theory of the deductive model of the pure sciences as a model for illustration means to design rules of inference and validation. In the following chapter, I will use nine categories of criteria to illustrate abstract concepts in practice through the visual arts. However, it is not always easy to distinguish between the current language and metalanguage when the latter is expressed in the current language. To distinguish them, I use colour as described below.

Secondly, I felt it was necessary to create from the colours used an ideography that could be applied to illustrate abstract concepts. For this, I was inspired by the work of Oliver Byrne (1847) who employs colours and geometric shapes to solve algebraic equations and geometric problems. In the same vein, Hervé Tullet (2017), artist and author of children's books, has also created symbols and concepts based on the use of colour and movement. For example, he proposes to move fictitiously with the finger circles of colours, red, blue, yellow, giving the impression that they emit sounds (oh!, whaaouhou, ahahahaha…). The concept, both tactile ('put your finger on the circle') and visual is interesting for illustrating sounds on paper (without any real sounds). It nevertheless requires a person who can read the text (oh!, whaaouhou, etc.) to a very young child. The idea (or concept) of associating colours here not with sounds but with reasoning, distinguishing between language and metalanguage, was not self-evident. I found by chance the work of the scientist George-Louis Le Sage and his idea of playing cards to write down his thoughts, problems and solutions. As mentioned above, his work gave me the idea to summarise the basics of logic on 140 cards and to classify its three arts (the art of definition, the art of judgement and the art of reasoning) using different colours. In trying not to turn these cards into a pure school textbook, a second idea that I had with Le Sage's cards was to use them, unlike him, as playing cards. However, moving from an educational textbook to a playful game through visual arts poses both theoretical and practical problems. This is the subject of the following chapter.

[100] Frege G., 1879. *Begriffsschrift, eine des arithmetischen nachgebildete Formelsprache des reinen Denkens*. Halle s/S: ed. Louis Nebert. Translated by C. Besson, preface J. Barnes, 1999, *Idéographie*, Paris: J. Vrin.

Chapter VI

The Art of thinking:
Illustration of games using visual arts

The reference model chosen is that of the deductive methods of classical and mathematical logic. Its use is the result of a story which takes place in two points (6.1 and 6.2). Then, in practice, I create two tools (6.5) to select criteria for illustrating abstract concepts through games using visual arts.

6.1 The story of the 35,000 playing cards of the scientist Georges-Louis Le Sage

In an attempt to solve the problem of classifying the 140 logic sheets I compiled during the summer of 2018, I was interested in the playing card of sociologist Jean-François Bert's book (I translate into English): 'How does a scientist think'? [101] The author presents the unpublished archives of the Genevan mathematician and physicist Georges-Louis Le Sage (1724–1803). Its archive consists of 35,000 playing cards, approximately 6 cm x 9 cm in size, and ordered in bags[102]. Each bag and the number of cards it contains are numbered. This material is both a laboratory of ideas and research on a multitude of various subjects and disciplines. It is a method of recording the scientist's ideas and those of the authors who inspired him. Also, it is an autobiography on the difficulties he encountered, in particular the problem of classifying a large amount of information (without a computer).
If I detected in Le Sage's playing cards a method that could help me organising and classifying the glossary I had compiled on logic, I paid attention not to end up in the same infinite spiral as he did. Le Sage's 35,000 cards, paradoxically are not used as playing cards and represent more of a drama for the scientist. He invested his life in numbering, labelling, grouping these cards, completing and constantly modifying them, for lack of a theory to classify them easily. As he never agreed to collaborate with others, he died with his disillusions on November 9, 1803, at the age of 79, leaving his 35,000 cards unpublished and unused[103]. However, the scientist Georges-Louis Le Sage was not the only scientist of his time to have used playing cards such as a card or notepad that was kept handy so as not to lose the new ideas that get inside his head. The card in the form of a playing card offers several advantages. They can be summarised in five points as follows.

[101] Bert, J. F., 2018. *Comment pense un savant ? Un physicien des Lumières et ses cartes à jouer.* Paris: Anamosa.
[102] This gave me the idea of classifying my 40 puzzles illustrating categorical Aristotelian syllogisms in bags of different colours.
[103] We find here the idea of Socrates, who in his maieutics went indefinitely from question to question, or of Plato in his method of dichotomies (or Divison A and non-A), which went *ad infinitum* from opposite ideas to opposite ideas (or from advantages to disadvantages, or from theses to antitheses), without being able to draw definitive conclusions, as Aristotle observed. Syllogism theory, where it is a question of drawing a definitive conclusion from two premises, has solved this endless problem. (Blanché, 1970, pp. 23–24).

TABLE 69

FIVE REASONS TO USE PLAYING CARDS AS A NOTEPAD

1. Notepad in the form of playing cards requires, first of all, conceiving a fair and distinct idea, concise and complete in a compact graphic space.

2. It unites ideas from one discipline to another.

3. It is a creative instrument for making discoveries, forging new ideas, new concepts and new theories.

4. Cards in the form of playing cards are all the more useful because they are easily transportable.

5. The abatement of the playing cards on the table makes it possible to arrange them and unite them by a sign, as in a puzzle, and to share them with others.

In short, the playing card can become a visual means of crossing ideas between art and science, and between pure sciences and fairy tales. However, there was still the problem of the classification of the cards to be solved. Of the 140 fact sheets and the glossary I had compiled on logic, this reinforced my idea that to understand the scientific theory, three main aspects were sufficient: first, the axioms and objectives, second, the rules of reasoning and judgement, and third, the results obtained, expected or unexpected. It can be summed up as a triptych of three ‘axioms-rules-result’ cards. To classify the 140 cards, I started by using colours: the axioms in red, the rules in yellow and the results in blue.

6.2 The creation of scholarly cards

The choice of the deductive reasoning model leads into distinguishing between language and metalanguage with a particular focus on metalanguage, i.e. on the axioms and rules which enable results to be obtained (conclusions in logic, theorems in mathematics). What should be highlighted employing the visual arts are the following points deduced from the previous developments.

TABLE 70

AXIOMS, RULES AND RESULTS IN A DEDUCTIVE MODEL

Symbolic notations:

1. Axioms and objectives (noted 1 a)

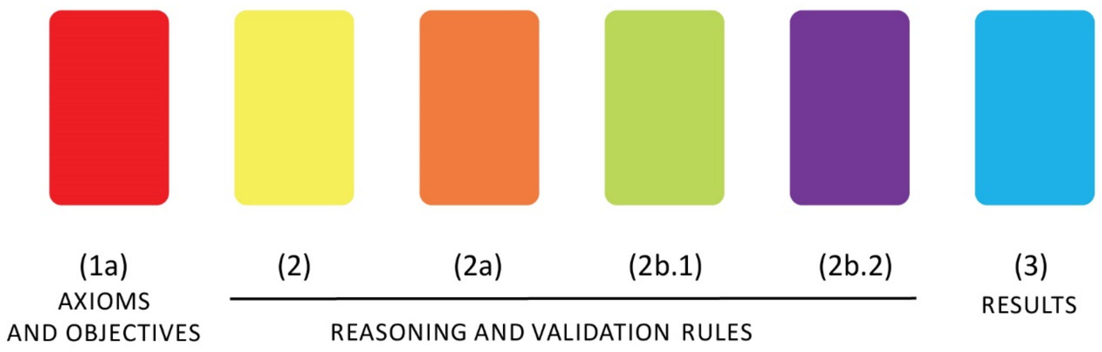
2. The rules which are of two kinds:
2a. rules of reasoning
2b. rules of verification and judgement (the instructions for using the rules) which are of two kinds and serve as criteria:
2b.1 to validate the reasoning rules
2b.2 to validate the results obtained according to the objectives and definitions (1 a) initially set.

3. The results (expected or unexpected) according to the objectives and concepts (1 a) initially set.

By convention and inspired by the colorimetric classification of the Maltese psychologist Edward De Bono and his *Six Thinking Hat* (1985), the primary colours will be restricted here to the axioms, rules instructions and results. Secondary colours will be employed to distinguish the three types of reasoning and validation rules.

TABLE 71

IDEOGRAPHY OF THE CONCEPTS USED IN THE DEDUCTIVE MODEL



This highlights a question. Although these six playing cards may suffice to illustrate the main concepts of the metalinguistic language of logic, they do not make it a game.

6.3 Illustrating abstract concepts through games

As Stéphane Chauvier (2007, pp. 93–97) points out, some see games everywhere. There are indeed several forms of games.

TABLE 72

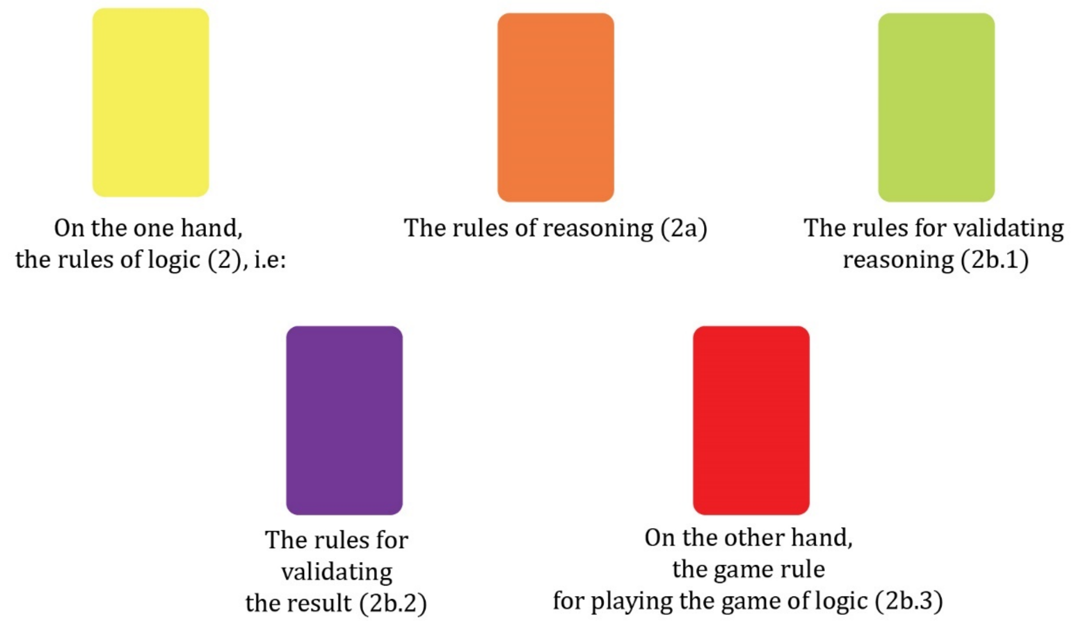
EXAMPLES OF DIFFERENT KINDS OF GAMES

. Institutional games: card games, draughts and chess, Go games, etc.;
. Language games: word games, crosswords' puzzles, role-playing games;
. Skill games: darts games;
. Team games: ball games;
. Mathematical and strategy games, including game theory with the notorious dilemma of the two prisoners, etc.

Despite this extensive list, not everything is a game and there is no game without rules, explains Chauvier. However, some rules are not necessarily the rules of the game. It is the case in logic where rules refer to reasoning and verification. In this way, if one wants to illustrate what logic is, it is necessary to distinguish, on the one hand, what concerns logic and, on the other hand, the rules of the game to play the Game of Logic. In other words, a game card must be added to the cards of the rules of logic.

TABLE 73

RULES OF THE GAME AND THE RULES OF LOGIC



This raises a new problem to solve. What is a game? As Chauvier (2007) points out, if we want to talk about a game, this concept must satisfy five conditions, criteria or rules, that I summarise and comment as follows:

TABLE 74

FIVE CONDITIONS TO MAKE A GAME

1st condition. It is necessary to have a material base: a board for chess, checkers, small horses; pawns, cards, figurines, etc. This will be the subject of different illustrations.

2nd condition. Game time and space are not the usual space and time. In other words, as in Einstein's general theory, there is a space-time of play that takes us into a dimension other than that of everyday life.

3rd condition. You have to set a goal to achieve which is the goal of the game and not a goal in life.

4th condition. The goal of the game must be attainable. But to arouse interest, there must be obstacles to overcome, as in any good scenario, according to Vladimir Propp's own thesis : what he calls the semiotics of the narrative.

5th condition. There must be a winner and a loser. However, the result is only valid in the context of the game. For example, I can beat all my competitors in the Monopoly game, but I won't be able to make use of this in my daily life. The 5th condition should in principle to exclude gambling: poker, roulette, lottery, etc. Because it is not just about fiction, since gambling can ruin or enrich the player.

Before wanting to illustrate Lewis Carroll's *Logic Game* and *Symbolic Logic*, one may wonder if they are games. Lewis Carroll's *Logic Game* for children is played with nine tokens, five grey and four reds incorporated in the book, which had to be moved to a cardboard representing two squares. The rules and combinations correspond to the 4 conditions mentioned above. However, as Jean Gattégno (2006, p. 13) writes, the game was of a particular kind since it was impossible to win or lose; which does not verify the 5th condition. *Symbolic logic* is no longer presented as a game, but as an exciting, useful and entertaining subject. In general, a game requires the creation of the rules of the game that can satisfy players. In fact, it is not easy to invent rules of play that allow many strategies such as those of chess.

6.4 Setting the rules of the game

There are games, such as Happy Families or the Battle game where the rules of the game are handed down from generation to generation. The purpose of my thesis is not to invent rules of games, so I will use well-known rules of games. According to Ockam's principle[104]: 'why make it complicated when you can make it simple', I am referring to the rules observed in many games. For example, in Game 4 – The Square of Opposition, I am inspired by the rules of the 'Battle game' for a 'syllogism-based battle'. However, what makes games interesting here are not the rules of the game that allow designating a winner and a loser, but the rules of logic that allow to reason and argue correctly. As in Lewis Carroll's *Game of Logic*, in Game 7, which I call The Robot, there are neither winners nor losers, but only players helping each other to solve logical puzzles. A scoring system is only used to qualify the difficulty of logical formulae solved using the game's counters. The games I have made take into account several criteria that are usually found, in all games. I have examined a large number of games (studies can be found in Bibliography II), in particular those that have survived over time. I have retained 21 criteria including age, the child's level of attention, the complexity or not of the game, the expected pedagogical objectives, the number of pieces, cards, size, and weight of the game, etc. Other researchers in psychology, marketing, pedagogy, etc. are free to add others. A list of some twenty-one criteria is included in the Booklets associated with the games.

[104] This principle attributed to the Franciscan monk Ockam (or Occam, circa 1285–1347) stipulates that, between several theories whose results are equivalent, we must choose the one that is the simplest, the most economic in terms of axioms, rules or assumptions. It is said that Napoleon asked the Marquis Pierre Simon de Laplace why his cosmology did not mention God. Laplace would have replied that he did not need this hypothesis. ESI, 2018, 'The Razor of Ockam', pp. 18–19.

TABLE 75

THE 9 BOOKLETS GENERALLY CONTAIN THE FOLLOWING ITEMS

1. Foreword	11. The winner
2. The aim of the game	12. The game's structure
3. Age level	13. Quick start
4. Number of players	14. How to play: detailed instructions
5. Number of cards in hand	15. How to proceed: card distribution...
6. Expected pedagogical outcome	16. The points score
7. Materials.	17. Strategy
8. The origin of the game	18. Game extensions
9. Time and space of the game	29. Illustrative examples, tests and answers
10. Goals to be achieved in the game	20. To Go Further: Literature and Works Cited

My discussions with publishers (Eyrolles) indicate that parents who consider their children to be more intelligent than average buy books and games for older age groups. To sell, since it is the parents who choose how to educate their child, the publisher can overestimate the age and underestimate the difficulties of understanding. For my part, and for the Lewis Carroll logic games that I have illustrated, I have indicated the ages that Lewis Carroll himself proposed or that can be found in publishing comparison for comparable games (Bibliography II). To visualise and materialise this question of age, I proposed to use a slide rule where the cursor varies according to the discussions between the author of the game, the illustrator, and the publisher. Moreover, since my prototypes have a pedagogical objective, as logic is part of the learning of pure sciences, the question of age was also, if not mainly, based on the national curriculum.
The material (item 7) indicates the number of pieces to play: playing cards, counters, figurines, dice, game board, puzzles, pop up, Logical Square, Venn and Lewis Carroll's diagrams, Truth Tables. A time limit (item 9) can be set at the start of the game so that it does not last too long especially with young children. In the Game 7.0 – 'Constructing a Logical Tale', the goals to be achieved (item 10) seek to awaken interest in pure, abstract, symbolic and formal sciences through the construction of short logical stories. Players will be able to create their own logical short stories in a simply and playfully way using dice and cards to test the consistency of their reasoning and the validity of their conclusion. A section called Quick start (item 13) can be used to play immediately (or almost immediately) with the aid of a few simple rules.
In conclusion, I obtain the following complete ideography which allows me to identify and differentiate by colours the concepts, rules and results, including the rules of the game to be illustrated by means of visual arts.

TABLE 76

COMPLETE IDEOGRAPHY OF CONCEPTS,
RULES AND RESULTS, INCLUDING THE RULES OF THE GAME

In the end, this ideography includes 7 cards, i.e. 6 cards and a game rule. By convention, primary colours will continue to represent axioms and objectives, rules and results. Secondary colours will be used to distinguish the reasoning and verification rules outlined above (2a, 2b.1, 2b.2). A warm tertiary colour, purple (red-purple), will be added to represent the rules of the game (2b.3). The instructions for operating the rules (the yellow card) will also be associated with the game rule. It will allow for the use of only six cards. However, given this large number of criteria, with, on the one hand, 21 items concerning games and, on the other hand, the nine categories of criteria concerning the illustration of reasoning, two questions arise. How in practice can these two kinds of criteria be selected and combined? For this purpose, I refer to two tools. The first is that of the slide rule, the second that of the colour circle.

6.5 Two tools to choose and combine evaluation criteria according to the objectives set

Choosing some criteria over others is ultimately defining criteria for judging. These judging criteria cannot be purely arbitrary. For example, in logic, the rules for validating the reasoning – which are judgement rules – should allow deciding whether reasoning is correct or not. In the same way, establishing game rules implies selecting criteria, for example, the age of the players. These are two different and complementary concepts. The first idea is to define, classify and combine criteria[105], the second idea is to make choices of combinations of criteria according to the objectives. The first tool, the slide rule, to which I am referring is used to establish correspondences between the criteria, the second, the colour wheel, is used here to combine them.

6.5.1 First tool: A graduated ruler

In children's books, as in games, one of the most frequently used criteria is age. It implicitly assumes a child who cannot read will only be able to look at images and handle objects. It assumes, being too young, he will only be capable of perceiving simple and primary colours. As a result, one might think young children will only be able to comprehend elementary and concrete concepts. Another way of conceiving things is precisely to use coloured objects and the possibility of handling them to make abstract and complex concepts understandable. This is what, for example, the artist Tullet (2017) does when he uses colour in his books for children, making them aware of sounds only through images. Here, the slide rule[106] allows establishing a relationship between several criteria. It consists of sliding a graduated ruler onto a fixed graduated ruler. By linking, for example, the age and the number of illustrations per book, it makes it possible to establish a precise exchange between the author, the illustrator and the publisher. The concept of graduation makes it possible to visually envisage a gradual learning of abstract concepts according to their difficulty of comprehension.

[105] The combination idea can be found in *De arte combinatorial* (1666), a book of youth by the German philosopher and scientist Gottfried Wilhelm Leibniz. Reprint 2012. *Dissertatio de arte combinatoria , in qua, ex arithmeticae fundamentis*. Paris: Hachette Livre, BNF.

[106] I am inspired here by the traditional slide rule. This beautiful object was originally designed as a circle (in 1630), then in its modern version since 1654 as a sliding ruler. This tool replaces complex multiplication and division with simple addition and subtraction. If a and b are two strictly positive real numbers and n is a non-zero natural integer, and the symbol ln denotes the neperian logarithm, the logarithm of a product a x b is equal to the logarithm of the sum a + b, i. e.: ln (ab) = ln (a) + ln (b) and for division: ln (a/b) = ln (a) – ln (b).

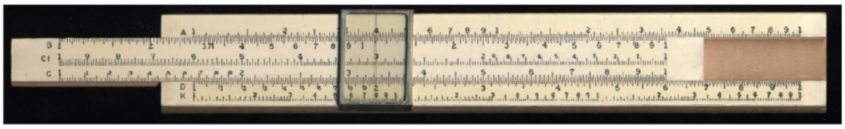
TABLE 77

THE SLIDE RULE

On the graduated fixed part, I propose to enter the image/text ratio in percentages, ranging from 100% image and 0% text to 100% text and 0% image.

While the age of the public concerned (e.g. 0 to 5 years, 6 to 10 years, etc.) will be indicated on the mobile part.

The two scales are brought closer together by moving the mobile ruler. This makes it possible to establish a direct link between the image/text ratio and the public concerned.



Several variants are possible. The fixed part can include several scales graduated according to the concerns of authors-illustrators-editors. For example, 1st scale, the image/text ratio; 2nd scale, the range of colours used and the degree of complexity of objects and games, with or without various sounds, with or without electronics, etc.; 3rd pedagogical scale, the level of difficulty in reading images and text, ranging from concrete to abstract concepts, 4th scale, the level of complexity of the techniques used. To visualise a sound and thus be able to illustrate it, it is usually associated with a word, image, colour or ideography, for example, music theory for music, Egyptian hieroglyphic writing, ideography for logic[107]. This slide rule can be materially realised and adapted to the selection of criteria (items) used in games or designed as a conceptual tool for reflection, just as in science one uses so-called 'thought experiments'.

6.5.2 Second tool: A chromatic circle inscribed in an Enneagon

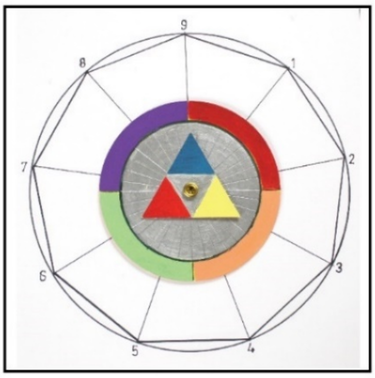
Firstly, to visualise axioms, rules and results, I use the concept of the colour circle which consists of combining primary, secondary and tertiary colours[108].

[107] Unlike the universally adopted music theory notation system, only a few traces of Frege's language remain in logic, for example the negation symbol '¬', the consequence symbol '⊢' or the tautology '⊨'. This shows that an abstract concept to be effective must be shared by as many people as possible. In *Symbolic Logic*, Lewis Carroll (1896, chap. III) adopted a personal ideography with logical symbols and charts to solve categorical syllogisms through a system of 'equations' and charts (a tree method). This system, which was not adopted by logicians is not examined in this thesis (Bartley, 1977, pp. 255–319).

[108] For the chromatic disc, I refer to *The Colour Wheel Company's disc*, [online] Available at: <www.colorwheelco.com> [Accessed 19 June 2020].

TABLE 78

THE CHROMATIC CIRCLE OF THE TRIPTYCH
'AXIOMS AND OBJECTIVES – RULES, RESULTS, AND VALIDATION RULES AND RESULTS'



In the centre, the primary colours of the triangle represent the primary concepts (axioms, postulates, definitions, hypotheses) and objectives in red, the rules in yellow and the results in blue. In the inner circle, secondary colours are used to distinguish the rules of reasoning and verification set out above: 2a in orange, the rules of reasoning; 2b.1 in green, the rules of judgement to validate the reasoning; 2b.2 in red-purple, the rules to validate the results and 2b.3 in red, the rules of the game. By design, the triad of the centre can rotate. It allows to think about several possible combinations between metalinguistic concepts and validation rules.

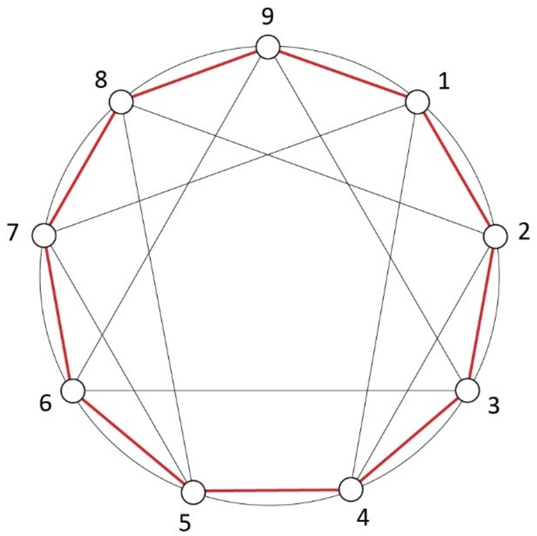
Secondly, I inscribe the colour circle in an Enneagon (neagon or enneagram) to visualise the nine categories of criteria (1. criterion of understanding, 2. 'image/text' fidelity criteria, 3. efficiency criteria, and so on). Enneagram[109] is originally an esoteric figure (in ancient Greek, ennea means nine). The numbers corresponding to the categories of criteria, from 1 to 9, are placed clockwise. In geometry, a neagon is a nine-sided polygon. The construction[110] here is the result of an association of ideas between the painters' chromatic circle and the geometric figure of an Enneagon.

[109] Enneagram can be used to present different human characters and combine them: point 1, the Perfectionist; point 2, the Altruist; point 3, the Battant, etc. See, for example, Palmer, H., 1995. *The Enneagram, in Love and Work*. San Francisco: Harper Collins Publisher Inc.

[110] It is possible to build a disc on which the 9 categories of criteria proposed will be inscribed on a circle surrounding a regular enneagon. i.e. a polygon with 9 vertices and 27 diagonals whose nine sides have the same length and whose angles in the centre have the same measure, 40 degrees. The 9 categories of criteria arranged on the enneagon are then associated with the chromatic circle, to constitute the final tool.

TABLE 79

THE 9 CATEGORIES OF CRITERIA ON THE ENNEAGON



The idea of using a chromatic circle to combine colours, here nine categories of criteria, is not new. Isaac Newton placed the colours of the visible spectrum in a circle, without classifying them, and rotated it fast enough to regain the sensation of white light[111]. From the middle of the 19th century onwards, colours were classified (primary, secondary, tertiary) and represented in order on a circle or disc[112]. The association of the chromatic circle with the Enneagon allows 9 categories of criteria to be linked to the 21 criteria (items) of the games. If one visually follows the direction of the arrows, the reading order will be 1, 4, 2, 8, 5, 7, 1 and then 3, 9, 6 which form a triangle. One will start by using the first category of criteria, for example, according to importance to comprehending a scientific text (axioms, rules, results) before illustrating it. Next, one will use the 4th category of criteria: the logic. The essential point here will be to avoid contradiction between the text and its illustration. Then, one can progress to the 2nd category of criteria: the image/text fidelity ratio, for example, to avoid misinterpretation, and so on. The triangle will draw particular attention to the following categories of criteria: 3 (efficiency and utility), 6 (creativity, innovation and discovery) and 9 (ethical aspect). This ultimate point placed at the apex of the triangle of the Enneagon will attach particular importance to ethical issues, especially when visual art concerns the education of young children. Others will prefer using the 9 categories of criteria in their natural order from 1 to 9.

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111 To build the coloured spinning tops with paper and a match, see: *Newton's Disc*. [Online] Available at: < https://www.123couleurs.fr/expériences/expériences-lumière/el-mélangestoupies/> [Accessed 19 June 2020]. For *The chromatic circle of painters*, see [online] Available at: < https://techniquedepeinture.com/les-secrets-des-maitres-les-couleurs/>[Accessed 19 June 2020].

112 I refer here to the Chromatic Disc published by The Colour Wheel Company, U.S., 2015. Oregon: Philomath, Artist's Mixing Guide Colour Wheel. [Online] Available at: <www.colorwheelco.com> [Accessed 19 June 2020].

6.6 The selection of criteria

To begin, I group the nine categories of criteria proposed into three groups.

TABLE 80

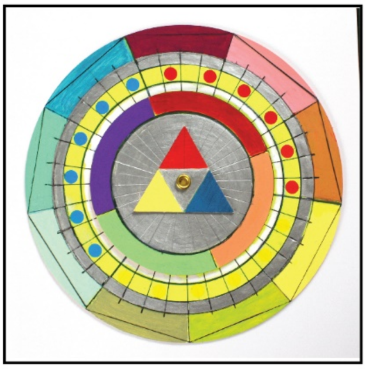
THE COMBINATION OF CRITERIA

Group 1. Criteria 1 to 3. Comprehension-fidelity-efficiency. It is represented on the enneagon by a gradation of three red colours (dark red, pink, ochre).

Group 2. Criteria 4 to 6. Logic-communication-creativity. It is represented by a gradation of three yellow-green colours.

Group 3. Criteria 7 to 9. originality-aesthetic-ethical. It is represented by a gradation of three blue-green colours.

The outer circle is divided into three equal parts. Each category of criteria is composed of several sub-criteria materialised by red, blue and yellow dots.



Then, the wheel can be turned to examine new combinations, as it is done on a chromatic circle.

The wheel can be turned to examine new combinations, as it is done on a chromatic circle.

6.7 The game of combinations

These conceptual tools aim to answer two practical questions. What needs to be specifically illustrated? Or in a broader conception, which visual arts tools should be used (drawings, cards, counters, game board, pop-up) to meet the criteria that will be selected?
The geometrical figure of the Enneagon associated with the chromatic circle highlights a first essential and more general question. How many combinations of criteria can be achieved by placing the 9 categories of criteria on the Enneagon in a different order? It is the same problem as knowing how many different ways to place 9 people around a table on 9 chairs. Or if one displays 9 paintings, how many variations are possible? The answer to this question is based on the combinatorial arithmetic theory (arrangements, permutations, combinations).

This is a crucial question in logic as well as in probability theory. In a formal and deductive approach, one must start by listing *all possible cases*. This concept of enumeration (combination) is used in Aristotelian logic to determine in its theory of syllogisms the amount of valid reasoning which can be constructed (Games 1 to 3). This counting concept will be found later in Boolean logic and in the Truth Tables highlighted by Wittgenstein in the *Tractatus logico-philosophicus* (1921), as shown in game 7.2. The combinatorial calculus will be developed in Pascal's Triangle (1654, published 1665), that has become nowadays, a full-fledged mathematical discipline with particular relevance to the calculation of probabilities.
From this perspective of combinatorial calculation, several questions arise. First of all, to test, validate, produce or judge an illustration or a game, is it necessary to use the proposed set of 9 categories of criteria and 21 items for games? Should they be operated 'successively', that is in a precise order, starting for example with the criteria of comprehension before the criteria of aesthetics or communication, or on the contrary, should they be applied simultaneously', that is in a non-ordered way, by choosing for example the criteria of aesthetics before those of logic and understanding? Secondly, whether it would be sufficient to choose only two or three categories of criteria? In this case, can the same criterion be repeated several times (e.g. 1,2,3; 1,3,4, etc.) or should it be used only once (1,2,3, 4,5,6, etc.)? Thirdly, is it possible to visualise the various possible combinations using visual diagrams?

6.7.1 How to select the nine categories of criteria proposed?

There are several tools to answer these questions. Firstly, the cross-table is a tool for creativity in the development of theories, including literary theories as Rabau and Pennanech (2016) have shown. It is an educational tool often used in children's quizzes and games. It is also used in enumeration theory. The theory is based on two concepts: order and repetition. Crossing the two criteria, order and repetition, gives four possibilities[113].

[113] I synthesise this presentation from several books, among others: Berrondo-Agrell, M. and Fourastié, J., 1998. *Le calcul des probabilités compréhensible pour tous, exercices avec corrigés*. Quebec: Gaëtan Morin Éditeur. (Probabilités — Serveur de mathématiques — LMRL, 2015. Analyse combinatoire. [online] Available at: http://mathematiques.lmrl.lu/Cours/Cours_1re/1B-probabilites_cours+exercices_.pdf [Accessed 12 August 2020]. Here, the objective is to draw attention to the concepts of order and repetition.

TABLE 81

CROSS TABLE OF ARRANGEMENTS, PERMUTATION AND POSSIBLE COMBINATIONS

Number of selected items: p Total number of items: n (with $p \leq n$)	Importance of order	Without regard to order
With repetition of items	Arrangement $L_n^p = n^p$	Combinations $D_n^p = C_{n+p-1}^p$
Without repetition of items	Arrangement $A_n^p = \frac{n!}{(n-p)!}$ Permutation (p = n) $A_n^n = n!$	Combinations $C_n^p = \frac{A_n^p}{p!}$ Or $C_n^p = \frac{n!}{(n-p)! \times p!}$

However, when the number of criteria to be taken into account becomes important (here 9 categories of criteria, 21 items for games), the visual method of cross tables becomes tedious. Another tool consists in using mathematical formulae that have already been demonstrated by mathematicians. The advantage and the mathematical beauty of these formulae is that, having been demonstrated, they are true whatever the number of elements taken into account. I give here two applications: the first to calculate the Aristotelian number of possible syllogisms, the second to combine the nine categories of criteria proposed.

TABLE 82

ARISTOTLE'S SYLLOGISTIC FIGURES

According to the 4 types of propositions (A, E, I, O) which constitute for each Figure I, II, III, IV categorical argumentation, the cross-tables have made it possible to establish in Aristotelian logic 256 possible models of reasoning (Games and Booklets 1 to 3). The enumeration is done for each of the 4 Figures with order and repetition of the 3 letters (AAA for the Barbara syllogism, EAE Celarent of Figure I, etc.). This corresponds to the general formulae:

$$L_n^p = n^p$$

If p = 3 and n = 4 then $L = 4^3 = 4 \times 4 \times 4 = 64$ and for 4 Figures: $64 \times 4 = 256$ possible syllogisms. It gives the result already calculated with the cross tables.

Mathematical formulae are applied here to enumerate the nine categories of criteria proposed to establish specifications on how to conceive illustrations and games through visual arts.

TABLE 83

COMBINATORIAL MATHEMATICAL FORMULAE

$$L_n^p = n^p$$

1. Applying this same formula to only 3 categories of criteria chosen among the 9 proposed gives 9^3 possibilities = 9 x 9 x 9 = 729 possible choices.

$$A_n^n = n!$$

2. If an author, an editor and an illustrator were to decide to establish an ideal relationship between a text and its illustration by using simultaneously (in order) and without repetition all of the 9 categories of criteria proposed (each criterion within a category may have nuances), 9-factorial arrangements would have to be considered. This is the application of the permutation formula (read factorial n)[114].

n! = 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 362,880 ways to place and choose the 9 categories of criteria on the Enneagon circle. For example, these 9 criteria can be placed either in the usual order 1, 2, 3, 4, etc., or in the order of the esoteric Enneagram 1, 4, 2, 8, etc. This number 362,880 is not infinite, but its application would be difficult. As in the Aristotelian model, which retains 24 valid syllogisms against 256, it is necessary to define rules and criteria to eliminate a large number of these possible choices.

3. If, in order to reduce the number of possibilities, each category of criteria is chosen only once, without any particular hierarchical order and without repeating it, the number of combinations will be given by the formula:

$$C_n^p = \frac{n!}{(n - p)! \times p!}$$

For p = 3 and n = 9, the number of possible choices[115] will be 84.

The last result (84 possible choices) can be obtained visually using Pascal's triangle which is another tool for calculating combinations.

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[114] In a general way and by convention n! (read factorial n) corresponds to multiplying all the numbers up to n. Let n! = 1 x 2 x 3 x 4 … x n. And by definition 0! = 1! = 1. Some children's puzzles or word games are based on this mathematical formula. How many ways can three letters A, E, T, be classified? Answer: AET, ATE, EAT, ETA, TAE, TEA. Here is the demonstration. The first letter can be any one of the three, the second, one of the two remaining letters, and the third, the one left over, i.e. 3 x 2 x 1 = 6 words. Or in general for n letters with n ≥ 2: n x (n – 1) x (n – 2)… 3 x 2 x 1 = n! (called factorial n). The most general formula of arrangements A_n^p responds to the following issue: How many possibilities are there to classify in order 3 horses (p = 3) among 10 at the start (p = 10)? Answer: (10!)/(10-3)! = 10 x 9 x 8 = 720 possibilities.

[115] Construction of Pascal's triangle: Starting from 1 to the first line, this is the initial line (n = 0). To determine the term in the next line, take the term just above it and add to it the one just before it (0 if there is nothing). Reading of the table: if p = 3 and n = 9 then the number of combinations is:

$$C = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = \frac{504}{6} = 84$$

TABLE 84

PASCAL'S TRIANGLE

	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

In this cross table, 84 is located at the intersection of column p = 3 and line n = 9. Choosing 4 categories of criteria (p = 4; n = 9) would lead to 126 possible choices as shown in Pascal's triangle.

6.7.2 The chosen criteria

A publisher could use only the commercial criterion (a criterion belonging to a category 5 of the proposed categories of criteria). To select illustrations (subjects, authors, illustrators), he could choose to use statistics and opinion polls. Without other rules and judging criteria, the problem of choice would not be completely solved. For it should be noted that the statistical theory of opinion polls is based on the calculation of probabilities which itself depends on the enumeration. Historically, Blaise Pascal, in his correspondence with Pierre de Fermat, developed the basis for the calculation of probabilities from gambling situations (Pascal, 1654). The concept of probability based on that of chance introduces other forms of logic than the one illustrated here, such as fuzzy logic. It is only briefly mentioned here because Venn diagrams are more often used to classify numbers or objects and calculate probabilities than to solve syllogisms. What the theory of combinations and the principle of an exhaustive enumeration of all possible cases show is that it is necessary for the practice to select a few criteria according to the objectives.

Choosing nine categories of criteria can lead to considering 362,880 possible combinations. To avoid finding myself in an unmanageable situation, I have retained only a few criteria. In all the games, I have favoured criteria belonging to groups 1 and 2 of the first four categories of criteria (understanding, image/text fidelity, efficiency, logical criteria of non-contradiction). Then I added specific criteria for groups 2 and 3, such as the age of the children for whom the games are intended and an ethical criterion such as child-friendly drawings and examples. As far as the logic is concerned, the difficulty of the games is progressive, from young children (puzzles 1 to 3) up to university level (Game 7: Truth Tables, Boole's algebra, the logic of propositions and predicates, computing science). Each criterion is ultimately an imposed or self-imposed constraint.

Conclusion and three examples

To conclude this chapter, and open a discussion, one may wonder whether it is not too constraining for an illustrator to impose as many constraints on himself as he would in the field of pure science and purely deductive reasoning models. As Graeme Sullivan (ed. 2010, p. 83) points out, there may consistently be a dilemma. On the one hand, the artist does not generally wish to lock himself into a precise methodology. His creativity, his imagination, his intuition must remain free. On the other hand, he aspires to a level of recognition, including in the field of research of his art. It requires at least one method, if not a methodology or theory as in the pure sciences[116], to be able to transmit it to other researchers. Without wishing to settle this debate, I can give three examples that show rules and constraints can be creative and produce interesting results.

A first example is '*Exercices de style*' (1947) – '*Exercises in style*' – by Raymond Queneau, co-founder of the literary group OuLiPo ('*Ouvroir de littérature potentielle*'). He tells the same story 99 times, in 99 different ways, which is an example of a literary constraint. The OuLiPo group, where we find among other Georges Perec, has brought together internationally literary, mathematical and scientific experts[117]. Perec's novel, *Life: A User's Manual'*[118], worthy of human comedy, traces the life of a building located at number 11 on the imaginary street Simon-Crubellier, in the 17th arrondissement of Paris, between 1875 and 1975. The facade was removed to show the inside of the rooms. The story evokes its inhabitants, the objects and lives that intertwine and that are followed from room to room in an unusual order.

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[116] Hannula, M., J. Suoranta and T. Vadén. 2005. *Artistic research. Theories, methods and practices*. Helsinki: University of Gothenburg, Sweden, also cited by Graeme Sullivan, 2010, p. 83. Mika Hannula explained 'that in theorising artistic research, it is always difficult to achieve consensus on purposes, methods, and practices… It is always going to be part of the inquiry process and that it is going to be continually questioned.'

[117] The OuLiPo is an association founded in 1960 by the mathematician François Le Lionnais with the cofounder the writer and poet Raymond Queneau, where we find, among others, Georges Perec.

[118] Perec, G., 1978. *La Vie mode d'emploi'* ('*Life: A User's Manual'*). Paris: Hachette, Livre de Poche.

The originality of the story and the method used lies in the constraints it imposes on itself: the 'polygraphy of the rider' or the 'algorithm of the rider'[119]. This choice of detail evokes Bruegel's *Children's games* (1560*), Netherlandish Proverbs* (120 proverbs, 1559) or the *Tower of Babel* (1563)[120]. In his 99 chapters with no less than 2000 characters, 420 constraints, Perec shows imagination and creativity. To achieve this, the author imposes rules and constraints on himself that seem to be the simplest and, paradoxically, the most assured way to be creative.

A second example is that of *The Library of Babel*[121] of the Argentine writer Jorge Luis Borges (1899–1986). In his prologue Borges cites the origin of the library, 'its history and prehistory': Leucippus, Lasswitz, Lewis Carroll and Aristotle. The short story speaks of a 'universe' as described by De Morgan, the British mathematician and logician. He refers to the term 'axioms' as in Euclid's Elements[122] and to a 'general theory' of the library that is exclusively based on notions of 'combinatory analysis'. It is his second axiom: 'There are twenty-five orthographic symbols[123].' For the opponents, this story is nonsense. Most of this combination of signs signifies nothing. The author replies that this opinion is fallacious because it is only a matter of cryptography and decoding over time.

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[119] In chess, the rider is usually represented by the head of a horse. The Knight moves 3 squares by drawing a capital L (2 horizontal squares + 1 vertical) or (1 vertical square + 2 horizontal). The problem of Perec's rider is that he must visit all the rooms of the building without going through the same one twice. This mathematical-logical problem is solved by an algorithm highlighted by the mathematician Leonhard Euler in a 1759, scientific study published in 1766. Diagrams and resolution: Wikipedia, 5 June 2020. *Knight's tour* [online] Available at: <https://en.wikipedia.org/wiki/Knight%27s_tour >[Accessed, 19 June 2020]. A doll's house would have given Georges Perec the idea of an exhaustive description of the parts of a building, the facade removed. Wikipedia, 3 October 2018. [online] Available at: <https://textualites.wordpress.com/2018/10/03/la-vie-mode-demploi-de-georges-perec/>[Accessed, 19 June 2020].

[120] The *Tower of Babel* is the title of several paintings by Pieter Brueghel the Elder painted after the biblical episode of the Tower of Babel. The large *Tower of Babel* (114 x 155 cm) was painted around 1563, the small *Tower of Babel* (94 x 74 cm) around 1568. These paintings inspired by the *Book of Genesis* (Genesis 11:1–9) would represent the dangers of human pride, but also the failure of rationality in the face of the divine. Brueghel the Elder, P., 1559–1563. Wikipedia, 24 April 2020: 1563. *Tower of Babel*. [online] Available at: <https://en.wikipedia.org/wiki/The_Tower_of_Babel_ (Bruegel)>, and 1559, *Netherlandish Proverbs*. Available at: <https://en.wikipedia.org/wiki/Netherlandish_Proverbs>, and 1560, *Children's games*. Available at: <https://en.wikipedia.org/wiki/Children%27s_Games_ (Bruegel)> [Accessed, 19 June 2020].

[121] *The Library of Babel* is inspired by a short story by the German writer, philosopher and mathematician Kurd Lasswitz entitled *The Universal Library* (1904). Borges, J.L.,1956. *Fiction*s. Translated from Spanish by A. Hurley, 2000, original title *Ficciónes*. London: Penguin Books, modern classics, pp. 65–74.

[122] Euclid, 300 BC. Reprint 1956. *The thirteen books of The Elements*. Translated with introduction and commentary by Sir Thomas L. Heath, vol. 1, books I and II, vol. 2, books III-IX, 2nd ed. unabridged, 2019. New York: Dover Publications, Inc.

[123] Borges gives this precision (ed. 2000, footnote 1, p. 67.): 'The original manuscript has neither number nor capital letters; punctuation is limited to the comma and the period. Those two marks, the space and the twenty-two letters of the alphabet represent the twenty-five sufficient symbols to constitute all 'that can be expressed, in every language'; that is, all present, past and future books. There are repeated letters and several hundred thousand almost perfect facsimiles that differ from the correct book only by a letter or comma'.

The oldest men employed a very different language from the one we speak presently. People might adopt a language tomorrow that we couldn't understand today. The author starts seeking for the book that contains all the books, 'a book that is the cipher and perfect compendium of all other books'. It transports us back to Russell's paradox about sets that contain themselves. To overcome the contradictions between an infinite library and limited combinations of signs or symbols, the author proposes this solution at the end of the story: 'The Library is unlimited but periodic. The same volumes are always repeated in the same disorder – which, repeated, becomes order: the Order.' Here, to be precise we have an example of the link between literary fiction and mathematics combinatorial analysis.

As a third and last example, I could add Edwin Abbott's novel *Flatland* (1884) which shows us how to move from the second dimension to the third and then to the fourth dimension[124]. All these authors have succeeded in deconstructing, reconstructing and 'decompartmentalising' literary, artistic and scientific disciplines. The members of OuLiPo define themselves as 'rats who build the labyrinth from which they propose to leave[125]'. By the game of combinations and constraints, they allow themselves the opportunity to create almost infinite concepts and rules. Lewis Carroll, with his 'scientific' tales, will have paved the way, Abbott, Borges, Perec and others followed. In addition to the 'demolition' and reconstruction of the vocabulary to which he has devoted himself throughout his works, what is interesting in Carrollian syllogisms is the way to find a solution, and not the solution itself. Which scientist could be interested in the Carrollian conclusions, 'Babies cannot manage crocodiles?' 'No heavy fish is unkind to children?' This way of finding the solution requires the use of a metalanguage composed of axioms, rules and abstract concepts. It is this metalanguage that I illustrated through visual arts. I found that the constraints I had imposed on myself through the 9 categories of criteria proposed, the 21 items of the games and their possible combinations finally helped me to obtain the results that are presented in the third and last part of this research.

[124] *Flatland* is an allegory, written pseudonymously by 'A Square' which the author, Edwin Abbott, gives life to geometric dimensions: point, line and surfaces. Abbott, E. A., 1884. *Flatland. A romance of many dimensions.* London: Seeley & Coin. Reprint 1992. New York: Dover Thrift Editions.

[125] This definition is attributed to Queneau, R., 2002, *Abrégé de littérature potentielle*. Paris : Mille et une nuit, p. 6. (I translated into English).

Part III

Main Results

Chapter VII. Nine Prototypes and illustrated Booklets

This research shows – which was far from obvious at the outset – that it is possible to establish a bridge between pure science and logical tales, visual art and art of thinking, and more generally, to establish in illustration a link between theory and practice, two *a priori* antagonistic universes[126]. In addition to the possibility of reinterpreting Carrollian nonsense and illustrating logic games, my research is part of an artistic movement (Visual thinking) that remains to be better qualified, on which I will be able, with other illustrators, to pursue research beyond the present thesis.

7.1 Conjunction of theory and practice

At the end of this research, I have obtained results that could be classified in Candy's Guide (2006) in the fields of practice and theory, conceptually called in illustration 'practice-based-research' and 'practice-led research', although the boundary between the two is not easily defined. As my objective being to illustrate abstract concepts and complex reasoning for children through visual arts, I was able to see the veracity of Graeme Sullivan's statement: it is better to start with effective methods, or as Kurt Lewin's states: 'Nothing is more practical than a good theory.' To create prototypes of pop-up games and the associated Booklets concerning visual reasoning, I had to research the main concepts, axioms and rules contained in the classical and modern theories of logic. This led me to distinguish in each theory between their current language and metalanguage. The main difficulty was to illustrate the metalanguage of logic (its axioms and rules). This language-metalanguage distinction is not specific to logic. In written and spoken language, metalanguage is what is more commonly called 'grammar' (definitions and rules). In sum, in this thesis, my approach consisted of illustrating the 'grammar' rules of logic. So, to use visual art as a 'meta-metalanguage', i.e. a language for illustrating other languages, I had to distinguish between the current language of illustration (what one draws and what one sees, i.e. the practice) and its metalanguage (its axioms and rules, i.e. the theory). What is remarkable is that this metalanguage can be common to several languages. It allowed me to establish a bridge between art and science, pure science and tales. I used this theoretical design as scaffolding to make logic game prototypes.

[126] Touchet, Demulier, Guimbail and Laupies, 2018 (about antagonistic theories).

Children and the players involved in the games are only concerned with the results obtained, that is the Games and the lessons they can draw from them and deepen with the Booklets. Of course, a child does not have to worry about this academic research that helped me to achieve them, and whose pedagogical aim was to introduce him or her to logic. As Wittgenstein writes at the end of the *Tracatus-logico-Philosophicus* (1922, ed. 2014, point 6.54, p. 89: 'He must, so to speak, throw away the ladder after he has climbed up it'. However, depending on the age of the children and their interest in logic, the objective from the very beginning of my research was to give them a thorough knowledge of the Art of thinking through play, and not superficial notions that they would not be able to use. This explains the many details and bibliographical references provided in the Booklets which can be used by university students.

7.1.1 First results based on the practice of creative visual arts

The first results obtained are an application and a contribution to the development of research in the field of 'practice-based research'. I progressed from drawing techniques using black ballpoint pen on paper to vectorial drawing with three-dimensional supports, such as the pop-up games (60 cm x 60 cm x 15 cm). This combination of drawing and paper architecture forced me to revisit, in particular, the laws and practice of perspective (Alberti, 1435, Andersen, 2007) and the techniques of drawing by hand. The construction of pop-ups is itself part of the development of a paper architecture whose rules are still experimental (case study, 2.6 above and appendices to the bibliography). All the pop-up games presented here are personal creations and illustrations.

7.1.2 Second results based on theoretical research: Another approach to the image/text ratio.

Secondly, to build the logic game prototypes, the two theoretical models I used are those of pure deductive sciences (mathematics and logic) and not of analogical or experimental sciences. The first basic model is that of the Aristotelian theory of categorical syllogism. The second standard model is that of compound syllogisms of Stoic origin. This allowed me to illustrate through the visual arts the three fundamental arts of logic, in the order in which Tricot (1928) classified them, namely:

1. Art of definition, classification and concepts.
2. Art of judgement.
3. Art of demonstrations, proof and calculation.

The term 'art' is used here in the sense of know-how and logic is defined as a tool (*Organon*). The deductive reasoning model is used to visually validate (or invalidate) reasoning within the defined framework of the logic of categorical and compound syllogisms. To bridge the gap between the visual arts and the art of thinking, I introduce pedagogically and playfully, as a 'third dimension' of the image/text ratio, the possibility of illustrating abstract concepts and complex reasoning by means of creative visual arts. To do so, the image/text ratio takes into consideration the illustration of the axioms, the rules and results that constitute what I designate as the 'metalanguage' of a scientific theory. This metalanguage is illustrated by various means provided by the visual arts: drawings, diagrams, puzzles, game boards, counters and figurines, pop-ups, as well as sentences, signs and symbols. The result is a contribution to the development of research in the field of 'practice-led-research', in the sense that this approach makes it possible to rediscover conceptions in which drawing is associated with reasoning, such as in Euclidean geometry, Alberti's perspective theory or Leonardo da Vinci's prototypes. This opens up new possibilities for research in illustration in fields other than logic where definitions, axioms and rules play a crucial role, in literature for example (with semantic and syntactic grammar rules) and even in the study of History of Art (with its rules of perspective, colour harmonies, different conceptual artistic movements: abstract and modern art, readymade concept, etc.). It is within the framework of these artistic movements that I inscribe what I call here the Thinking Art, defined below.

7.2 Application of theoretical and practical research for the achievement of nine Prototypes and illustrated Booklets

The nine Pop up Games and Booklets retrace, employing the visual arts, the theory of logic, from antiquity to computer science, under the particular viewpoint of its main metalinguistic concepts (axioms and rules). The evolution of logic is divided into two main branches of activity: on the one hand, questioning, argumentation and discourse (Socrates, Plato, Aristotle, the Stoics), on the other hand, Truth Tables, Boolean logic and visual calculus of syllogisms. Game 7.0 bridges the gap between logic and storytelling by allowing players to create their own story based on Propp's narrative theory, Greimas' actantial model and semiotic square derived from the Medieval Logical Square of Opposition. Using visual arts, players will be able to test the validity of several reasoning models.
From a pedagogical point of view, the introduction to logic is done progressively in games and Booklets 1 to 4, first with the use of jigsaw puzzles, playing cards for the study of the Square of opposition (game 4), then with Euler, Venn (Game 5) and Lewis Carroll (Game 6) diagrams, starting with the basic model of the Aristotelian theory of categorical syllogism. The compound syllogism reasoning model is then examined (Game 7), using Boole's logic, Truth Tables and the principle of Natural deduction.

Summarised in the appendix are to be found 26 tables (7.2.1 to 7.2.9) which detail the different technical points that enabled me to write the Booklets as an instruction manual on formal logic and to present this complex and abstract discipline in the form of illustrated games. In the Booklets, the learning process is progressive, with many examples, looked at from different perspectives, and a specific bibliography to allow for further study. These include the use of the previously discussed chromatic circle, Truth Tables, Aristotelian rules for the validation of categorical reasoning, rules for compound syllogisms, the practical application of Boolean Truth Tables to electrical and electronic diagrams, and an incursion into the world of coding, robotisation and computer science. The following point (7.3) lists the Booklets, Pop Ups, puzzles and game boards.

7.3 Results: the creation of Booklets, Pop Ups, puzzles and game boards

First of all, to introduce logic and the art of reasoning, I used as a game the principle of building jigsaw puzzles which, for young children, seems to me easier to realise and understand than the diagrams of Venn (Game 5) and Lewis Carroll (Game 6). Then, to make logic as entertaining as possible, Game 7 allows players to create their own story or tale based on the categorical syllogism model. They can establish a link between Aristotle's logic and that of the Stoics and create compound syllogisms (hypothetical syllogism, dilemmas, contradictory arguments, *reasoning by the absurd*, etc.). The use of Truth Tables and coding allow the players a more mechanical reasoning approach to Boolean logic and the science of computers.

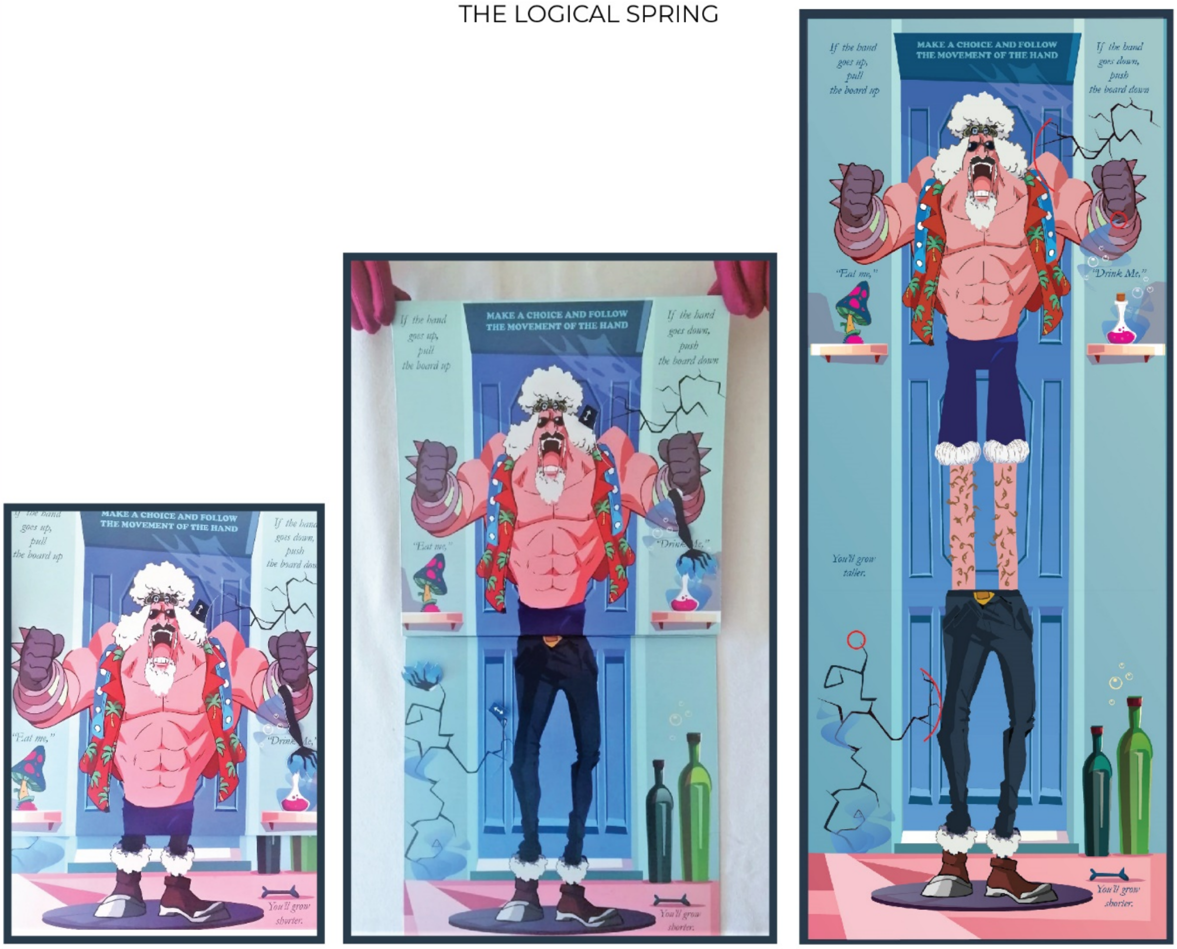
7.3.1 Lewis Carroll's dilemmas and paradoxes: two preliminary pop-ups

1. The Logical Spring

To make explicit the logic that is contained in Lewis Carroll's best-known tales, *Alice's Adventures Under Ground* (1864) and *Alice's Adventures in Wonderland* (1865), I made two preliminary pop-ups entitled: *The Logical Spring* and *The Cheshire Cat Paradox.* The first pop-up, *The Logical Spring*, highlights one of the five valid reasoning models identified by the Stoic philosopher Chrysippus of Soli (c. 280-207 BC). This valid argument known under the Latin name of *modus ponens* can be put and solved in Game 7 in the form of a logical equation: ((p => q). p) => q.

PERSONAL ILLUSTRATIONS

THE LOGICAL SPRING



2. The Cheshire Cat Paradox

The second preliminary pop-up entitled *The Cheshire Cat Paradox*, based on the implicit example given by Lewis Carroll in *Alice's Adventures in Wonderland* (chapter VIII), highlights the theory of paradoxes in which language and metalanguage are abnormally confused. Because of this confusion of languages, it is difficult to get out of a paradox as Lewis Carroll illustrates in *What the Tortoise Said to Achilles* (1894) or as in the *Crisis in the Foundations of Mathematics* stigmatised by Russell's Barber's Paradox. Game 7.1 allows setting paradoxes in the form of logical equations using a game board.

PERSONAL ILLUSTRATIONS

THE CHESHIRE CAT PARADOX

SYLLOGISM BATTLE

‘The executioner’s argument was, that you couldn’t cut off a head unless there was a body to cut it off from: that he had never had to do such a thing before, and he wasn’t going to begin at his time of life.

The King’s argument was, that anything that had a head could be beheaded, and that you weren’t to talk nonsense.

The Queen’s argument was, that if something wasn’t done about it in less than no time she’d have everybody executed, all round. (It was this last remark that had made the whole party look so grave and anxious.)’

Lewis Carroll, ed. 2006, p. 83

7.3.2 Jigsaw puzzle games 1 to 3

PERSONAL ILLUSTRATIONS

PUZZLE GAMES 1 TO 3

The learning of logic is done progressively through games. Lewis Carroll considered that *The Game of Logic* was accessible to school-age children[127]. The easiest puzzle games (1 to 3) I have designed in the form of wooden construction puzzles are for children aged 5 and over. The other Games (5, 6 and 7) with Venn and Lewis Carroll diagrams and Truth Tables are for older children aged 8 to 11 + years and for students interested in logic and pure science. Game 4 establishes a transition between the puzzles and the diagrams of Venn and Lewis Carroll. Game publishers could set age limits differently depending on their clientele.

[127] Gattégno, J. and E. Coumet, 1966. *Lewis Carroll. Logique sans peine*. Illustrated by Max Ernst. Reprint 2006, 6th ed. Paris: Hermann, pp. 12–13.

The purpose of the puzzles is twofold. Firstly, it is to make children aware that there are correct reasoning and incorrect reasoning. Secondly, it is to show them that the possibility of making puns by inverting the subject and the predicate in a sentence does not necessarily lead to correct reasoning; what logic calls imperfect reasoning. Booklet I is an introduction to logic, Booklets 2 and 3 show how to play with words and sentences to reduce complex arguments into simpler and more convincing ones.

GAME 1, 2 AND 3

PUZZLES 1, 2 AND 3: A VISUAL RESOLUTION OF STANDARD COMPLEX AND IMPERFECT SYLLOGISMS

Puzzles 1. Through the Logic Looking-Glass. (5+ Years.)
The key used here to attach one piece of the puzzle to another determines a unique and valid conclusion. The placement of the pieces of the puzzle, the colours and characters used, make it visually possible to discover that there are correct and incorrect reasoning.

Puzzles 2. The Mirror Game. (11 + Years.)
The objective is to reduce complex reasoning to equivalent simpler ones and to thwart fallacious reasoning. These puzzles combine words and arguments to deduce valid conclusions by adopting the same colour code as used in the puzzles games 1 and by using attributes such as a flower, a hat, blond hair, etc., which allows to easily memorise and manipulate valid models of reasoning.

Puzzles 3. The Mirror Game II on terms and propositions. (11 + Years.)
Game 3 allows playing with terms (subject and predicate) and propositions logically. By assembling the puzzles, in the manner of 'Where's Wally', and paying attention to the illustration, it becomes visually possible to carry out logical conversions, obversions, contradictions, dilemmas and paradoxes with the words.

7.3.3 Pop up game 4. The Square of Opposition Battle

Studying Lewis Carroll's Logical Tales led me to teach art history and drawing at secondary level, using the reflections developed in my thesis to structure my lesson plans. For example, Game 4 "The Square of Opposite Battle" is used in the classroom to teach children to develop argumentation from their knowledge. To do so, I present a painting, and instead of lecturing about the paintings, the game is used to organise debates with the pupils, allowing them to develop a logical argumentation to understand the story, the meaning of the painting and its technique.

While the "container" (the rules of logic and the rules of the game) remains the same, the content (the images and forms of the prototypes) can be altered. As a result, all the prototypes can be seen as the chess game where the rules of the game and the function of the pieces are important, while the look or the application of an image onto each piece can be adapted by other image makers. In the classroom, Game 4 was replaced by a Robot-game, built and illustrated by the children. The latter becomes a cardboard art object: 200 cm x 100 cm that the children appropriate, more visible and usable in class of 30 pupils than the pop-ups I presented in this research.

PERSONAL ILLUSTRATIONS

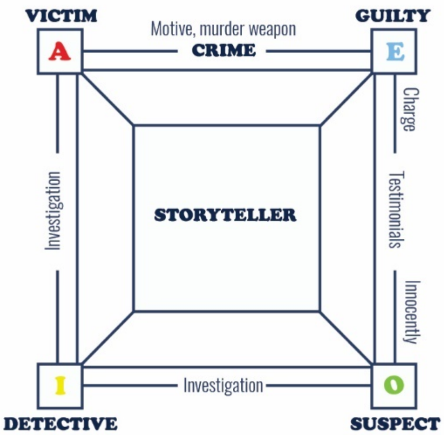
THE SQUARE OF OPPOSITION BATTLE

For this research the game 4 is illustrated in the form of a pop-up (60 cm x 60 cm x 15 cm) with a game board, cards, dice, counters, figurines and an illustrated instruction manual, Booklet 4. It is an introduction to the traditional Logical Square, also called Square of Opposition. Conceived as a game of verbal jousting where one player asserts something that another seeks to refute, it highlights contrary and contradictory propositions and subaltern propositions. Players will visually discover the basic rules of correct reasoning: non-contradiction, non-deduction of a generality from a particular case. The diagram is used in Game 7 to create, for example, short detective stories.

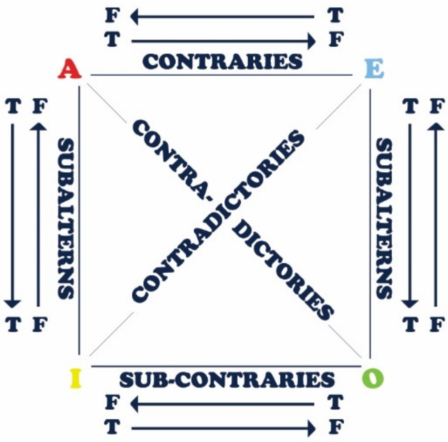
GAME 4

THE SQUARE OF OPPOSITION AND ITS APPLICATIONS

The use of this diagram shows that there are three practical ways to refute an argument: contradiction, counter-example, contrary proposition. The first two are the most important. A contrary proposition or an opinion judgement rarely replaces a mathematical demonstration. For example, if Goldbach's conjecture cannot be demonstrated mathematically, it is not enough for some people to think the conjecture is true while others consider it false. It is a judgement of opinion, and both statements can be false. This can lead to a dialogue of the deaf, with no possible conclusion.



A detective story - Greimas's semiotic square



Square of Opposition

7.3.4 Pop up games 5 and 6: Venn and Lewis Carroll diagrams

PERSONAL ILLUSTRATIONS

VENN AND LEWIS CARROLL DIAGRAMS

The purpose of these games is to visually solve categorical syllogisms including the syllogisms solved in the puzzle games 1 to 3. The Venn Diagrams Game 5 includes a pop-up game (60 cm x 60 cm x 15 cm), a game board, 24 cards, 46 wooden pieces and an illustrated instruction manual, Booklet 5, to use the rules of logic and the rules of the game, with 24 examples of valid syllogisms from Figures I to IV and 9 examples of fallacious reasoning. Through the 'Universe of discourse', players will discover the three arts of logic: the Art of definition, classification and concept, the Art of judgement, the Art of demonstrations, proof and calculation. They will be able to test and visually validate the conclusion of categorical reasoning (syllogism or sorite).

GAME 5
PERSONAL ILLUSTRATIONS
VENN DIAGRAMS

OBJECTIVES

The purpose of Venn diagrams is to determine the valid conclusion inferred from two premises (declarative sentences).

All animals that drink milk when they are babies are mammals;
All people are animals that drink milk when they are babies.
Thus, what conclusion can be deduced. Is the conclusion valid?

The validity of the reasoning and conclusion does not depend on the content of the premises. They are implicitly assumed to be true. Therefore, no importance should be attached to the content of the selected examples, which are only a pretext to distinguish between valid and invalid reasoning.

THE GUIDEBOOK

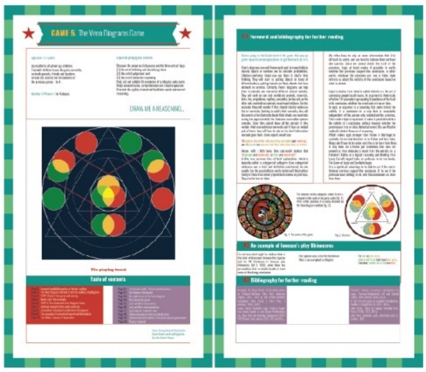
A booklet associated with the game explains how to obtain the answer handling the cards and the counters on the board. The valid conclusion is formulated on the back of each game card. Here, the conclusion is *All* S is P, or in conventional language: *All* people are mammals.

THE RESULT

The propositions are coded and illustrated as follows where S denotes the subject of the sentence and P the predicate:

A. Universal Affirmative: *All* S is P; e.g. *All* people are mammals.
E. Universal Negative: *No* S is P; *No* people are mammals.
I. Particular Affirmative: *Some* S is P; *Some* people are mammals.
O. Particular Negative, *Some* S is not P; *Some* people are not mammals.

This visual system highlights the principle of contradiction: *All* S is P and *Some* S is not P, as *No* S is P and *Some* S is P are contradictory propositions. It also recalls an essential logical principle: from a particular case '*Some* S is P', it is not possible to deduce that '*All* S is P'. Any generalisation from a particular case is an error of reasoning that should be avoided.



The Lewis Carroll diagrams Game 6 includes a pop-up game (60 cm x 60 cm x 15 cm), a game board, playing cards, counters and an illustrated instruction manual, Booklet 6. This game is used to visually solve categorical syllogisms. Players will be able to play with words and make the difference between the truth of a discourse's arguments and the validity of its reasoning. They will have the opportunity to use with words and sentences a very special addition table. Booklet 6 explains step by step the method, the rules of the game and the rules of logic and provides many examples of application.

GAME 6
PERSONAL ILLUSTRATIONS
THE LEWIS CARROLL DIAGRAMS

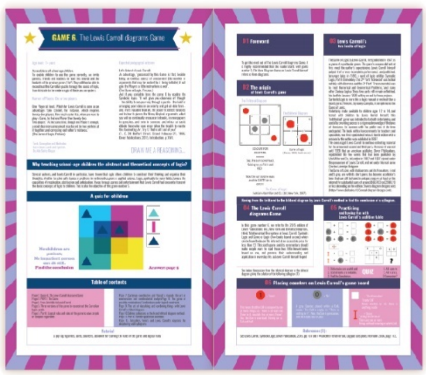
OBJECTIVES

The purpose of Lewis Carrolls' diagrams is to determine the valid conclusion that can be inferred from two premises, as in his example (Lewis Carroll, Dover publication, 2015, p. 62):

All diligent students are successful;
All ignorant student are unsuccessful.
Thus, what conclusion can be deduced? Is the conclusion valid?

THE GUIDEBOOK

A booklet associated with the game explains how to obtain the answer handling the cards and the counters on the board. The valid conclusion is formulated on the back of each game card.



THE RESULT

Lewis Carroll Instructions

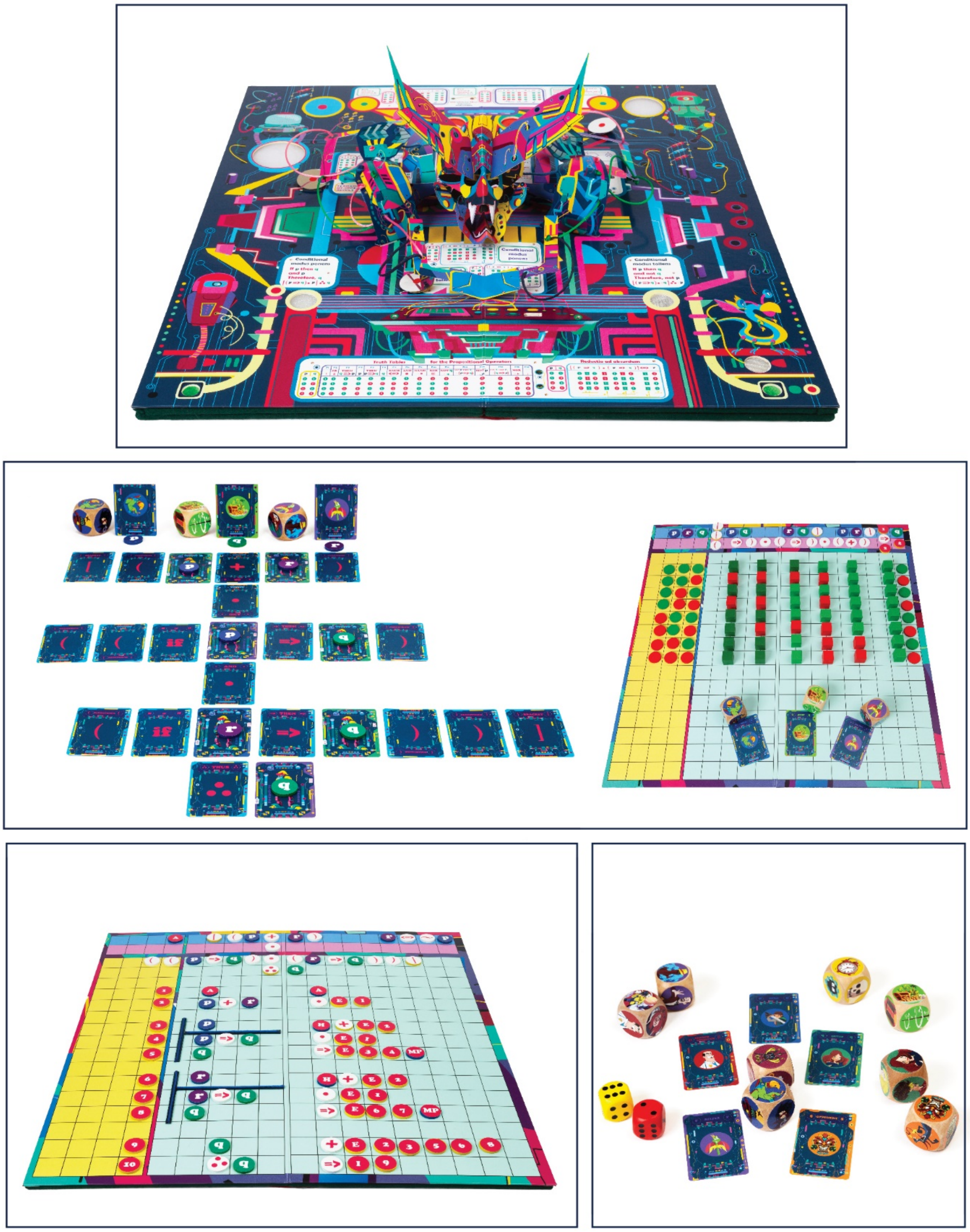
Let "the universe of discourse" be "students"; m = successful; x = diligent; y = ignorant.

The first coded proposition "All diligent students are successful", which Lewis Carroll divides into two propositions that retain the same meaning, is transcribed in trilateral diagram 1, and the second proposition "All ignorant students are unsuccessful" is transcribed in diagram 2. Here, for more clarity, S represents the subject (S = x = diligent students) and P the predicate (P = y = ignorant students).

7.3.5 Pop up game 7: The Robot

PERSONAL ILLUSTRATIONS

THE ROBOT



Game 7 consists of a pop-up, two game boards, dice and illustrated counters. It establishes different bridges, between the logic of Aristotle and that of the Stoics, between logic and storytelling, logic and *'The Electricity Fairy'*[128] of the physical sciences, logic and the computer sciences. It illustrates through visual arts and games two main themes: the history of logic, from Aristotle to modern computer logic, and its concrete and modern applications. Where historians rightly saw at the turn of the 20th century a point of rupture between philosophical logic and mathematical logic, the game 7 establishes through visual arts a bridge between the traditional Art of Thinking and the modern 'Art of Calculus'. The link between the two logic is established by the use of four games: 7.0,7.1,7.2,7.3.

Game 7.0: Logic and storytelling

The 'Construction of a Logical Tale' links logic and storytelling. The story can be based on the analysis of the Russian folklorist Vladimir Propp, the actantial scheme and Semiotic Square of the Lithuanian linguist and semiotician Algirdas Julien Greimas and the most frequently used narrative schemes. The 16 operators of the Truth Tables (and, or, implies, etc.) give players the possibility to create short logical stories in a simple and playful way using dice and playing cards with the sender, receiver, quest objectives, heroes, helper, enemy, and rival. Then players can test the consistency of their reasoning and the validity of their conclusion.

[128] La Fée Électricité ("The Electricity Fairy") is an "Art Deco" painting by Raoul Dufy for the 1937 International Exhibition in Paris, "devoted to art and technology in modern life". The composition takes place on 250 panels (H x W: 1000 cm x 6000 cm) from right to left, on two main themes: the history of electricity and its applications, from the first observations to its most modern technical applications. Paris: Museum of Modern Art.

GAME 7.0
PERSONAL ILLUSTRATIONS
THE ROBOT

APPLICATION

The eight dice make it possible to constitute a vast number of stories.

The sender (King, Queen, Princess, or other) entrusts a mission to the hero. The hero will have to overcome several obstacles (usually three obstacles). In his quest, the hero will be seconded by the intervention of allies, auxiliaries, or extraordinary objects or powers. The story informs how the hero confronts his opponent, enemies and untrustworthy hero. The failure of the opponents leads to the final situation: successful mission, hero victory, reward, happy ending, return to balance. According to the story, the hero's situation gets better, or it may get worse. Greimas's semiotic square makes it possible to conceive short detective stories as in game IV of the Square of Opposition.

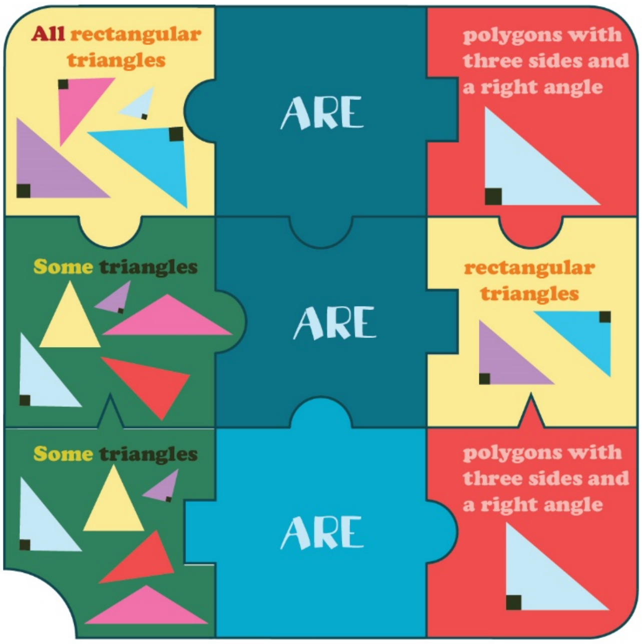
Game 7.1: The logic of categorical reasoning

GAME 7.1
PERSONAL ILLUSTRATIONS
THE ROBOT

In this game, using coloured counters, players translate categorical syllogisms into a logical equation to test its validity. This game allows memorising with the help of the coloured counters, the logical rules they will use to demonstrate the validity or invalidity of a conclusion. For example, the game allows checking the validity of the Darii syllogism, using coloured counters and a game board. By referring to the rules defined by Aristotle and specified in the corresponding Booklet, it helps to understand why this categorical syllogistic reasoning is valid. This makes it possible to translate the puzzles into a logical equation.

PERSONAL ILLUSTRATIONS

LOGICAL EQUATION OF PROPOSITIONS



AII1 Darii

All rectangular triangles are polygons with three sides and a right angle (90°);
Some triangles are rectangular triangles.
Some triangles are polygons with three sides and a right angle (90°).

or put in a symbolic form

All M is P
and some S is M
Therefore, some S is P

The logical equations of the valid syllogism Darii (AII1) can be written as follows.
1: p(A) . q(I) ∴ r(I)

Or, with the coloured counters:

 major premise  minor premise  conclusion

 valid conclusion  and

Each proposition (p, q, r) can take one of the four forms A, E, I, O.

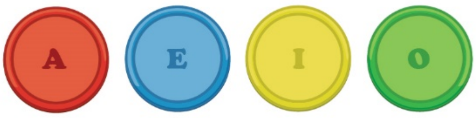


Figure I: Valid syllogism Darii (AII1)

1 

Game 7.2: Storytelling and compound syllogisms

GAME 7.2
PERSONAL ILLUSTRATIONS
THE ROBOT

This game illustrates the logic of propositions by proposing to learn and use the Truth Tables as we learn and use in arithmetic, the addition and multiplication tables. The game gives a brief introduction to the predicate logic. By referring to Boole's logic, it opens the way to computational logic and computer science.

As shown in preliminaries Games 1 and 2, there are other forms of syllogisms that can be used to construct short stories, such as Lewis Carroll does implicitly in *Alice's Aventures Under Ground* (reproduced in *Alice's Adventures in Wonderland*). Compound syllogisms of Stoic origin make it possible to construct more varied stories or discourse. The introduction of Truth Tables in Wittgenstein's *Tractatus Logico-Philosophicus* (propositions 4.31, 4.442, and 5.101) will mostly reduce the theory of the compound syllogism and the philosophy which were hidden in the calculation of propositions.

As Blanché (1970, p. 350) observes, in this modern logic, there is no longer any need to refer to axioms or rules such as Aristotle's. Truth Tables are to computational logic what addition and multiplication tables are to arithmetic. The aim of the game 7.2 is to allow players to use Truth Tables to validate compound syllogisms. Concerning the application of a short story or a tale, whatever the version (7.1,7.2,7.3), the game takes place in three steps as indicated in the Booklet 7.0 under Quick Start.

QUICK START

TO CREATE A SHORT LOGICAL STORY

Firstly, the first player starts by constructing a short story using illustrated dice (senders, receivers, objectives of the quest that mobilise heroes or heroines, adjuvants or helpers, opponents, enemies, rivals, villains or aggressors who produce mischief).
Secondly, using the illustrated dice and some of 100 playing cards allowing to use 16 logical operators, the player writes his story in symbolic and coded form. Then, he transcribes the tale into a logical equation by linking the propositions by means of logical operators (and, or, implies, etc.) and using the Boolean binary language composed only of 0 and 1. This makes it possible to use special addition tables (detailed in Booklet 7.2).
Thirdly, using the truth tables (Game 7.2) or the Natural Deduction (Game 7.3) with counters on a game board, the second player, with the help of the first, tests the consistency of the story and the validity of its conclusion.

Players will be able to construct short stories based on the following nine thinking models given as examples:

1. *Modus ponens*: $((p \Rightarrow q) . p) \therefore q$
2. *Modus tollens*: $((p \Rightarrow q) . \sim q) \therefore \sim p$
3. Hypothetical syllogism: $((p \Rightarrow q) . (q \Rightarrow r)) \therefore p \Rightarrow r$
4. Negation of the conjunction: $(\sim(p . q) . p) \therefore \sim q$
5. Exclusive disjunction: $((p \oplus q) . p) \therefore \sim q$
6. Non-exclusive disjunction: $(p + q) . \sim q) \therefore p$
7. Dilemmas (constructive and destructive)
8. Contradiction
9. *Reductio ad absurdum*

The scoring of point makes it possible to measure the difficulty of the syllogisms solved. It simply consists of counting the number of variables (p, q, r, etc.) and operators (and, or, implies, etc.) used.

For example, in the game 7.2, players can verify the validity of a complex constructive dilemma by means of a game board and coloured counters, using true tables. Booklet 7.2 gives an example of Corax dilemma. According to legend, the Greek Corax of Syracuse, in ancient Greek Κόραξ (6th century B.C. - 467 B.C.) was a sophist, and founder of rhetoric. He taught the art of persuasion and claimed that he could demonstrate everything and its opposite. He allegedly asked his student Tisias to be paid for his teaching on the only condition that Tisias would win his first trial. Otherwise, he would not ask for anything because that would prove the inefficiency of his method[129].

[129] Couillaud, B., 2003. *Raisonner en vérité*, Paris : François-Xavier de Guibert, Paris, p. 427. This example is also cited in Siu-Fan Lee (2017, pp. 246–247).

AN EXAMPLE

CORAX'S DILEMMA

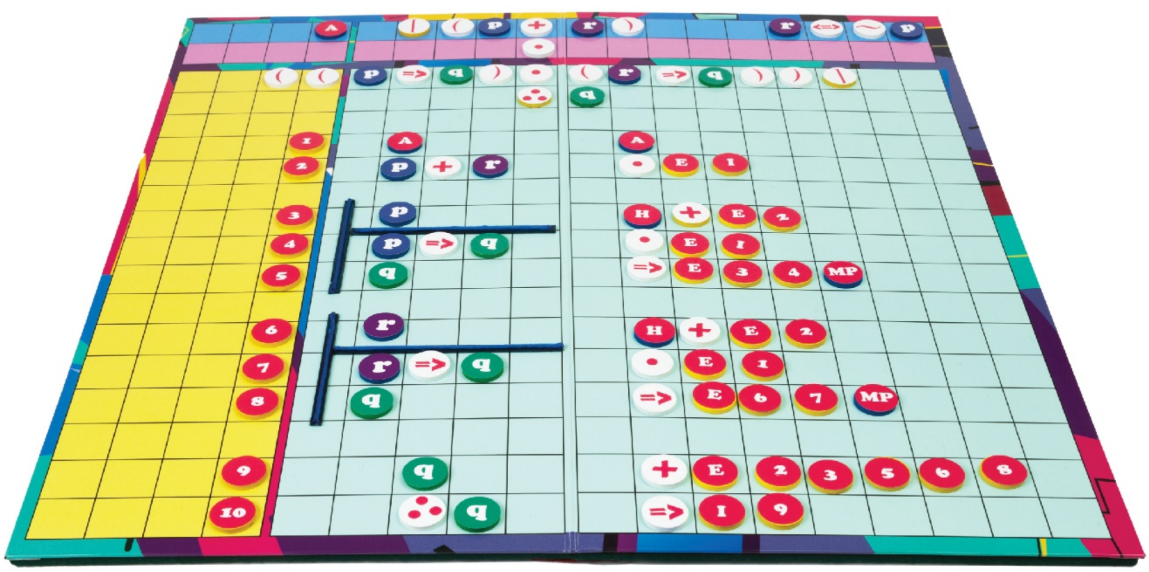
Tisias either wins (p) or loses (r = not-p)	p + r
If he wins, according to the contract he has to pay (q)	p => q
if he loses his trial, he has to pay (q)	r => q
In either cas, he has to pay (q)	Therefore, q

This constructive dilemma is written:
[(p + r) . ((if p => q) . (if r=> q))] ∴ q

It can be written more simply (Peeters and Richard, 2009, pp. 99–100), if p = he wons, and not-p = he loses, noted ~p:
[(p + ~p) . ((if p => q) . (if ~p => q))] ∴ q

Game 7.3: Natural deduction

GAME 7.3
PERSONAL ILLUSTRATIONS
THE ROBOT



This game proposes to use the logic of Natural deduction to test the validity of imagined logical tales by highlighting dilemmas, nonsense and contradiction. The Booklet 7.3 explains how to use the Natural deduction method using counters and a game board.
The following example uses Corax's dilemma, the validity of which can be demonstrated by this method. According to the rules of the game, solving the dilemma yields 7 points (2 variables, p and q, 1 negation sign, 3 connectors: and, or, implies and 1 final sign for the valid conclusion, parentheses and brackets are not counted).

GAME 7.3

CORAX'S DILEMMA DEMONSTRATED BY THE NATURAL DEDUCTION METHOD

The formula for Corax's dilemma is written here with the symbols, cards and counters of the game.

[(p + ~p) . ((if p => q) . (if ~p => q))] ∴ q

The demonstration is as follows:

1	[(p + ~p) . ((if p => q) . (if ~p => q))]	^ (Assumption if…)
2	p + ~p	. elimination, 1
3	p	Hypothesis (H, + elimination, 2)
4	p => q	. elimination, 1
5	q	=> elimination, 3, 4 MP (modus ponenes)
6	~p	Hypothesis (H, + elimination, 2)
7	~p => q	. elimination, 1
8	q	=> elimination, 6, 7 MP (modus ponenes)
9	q	+ elimination, 2, 3-5, 6-8 (if 1 then 9)
10	[(p + ~p) . ((if p => q) . (if ~p => q)) => q	=> introduction, 1, 9

Point 10 concludes that the dilemma is valid

In sum, this 'Calculus logic' is what Tricot (1928, ed. 1973, pp. 305–314) and Blanché (1970, p. 351) call 'logicist reduction'. Logic becomes a mechanical science of calculation. Truth Tables are learned mechanically by reciting them by heart, such as multiplication tables in arithmetic, without reference to metalanguage (i.e. the axioms and rules of arithmetic). If this metalanguage had really disappeared, there wouldn't have been much to illustrate here.

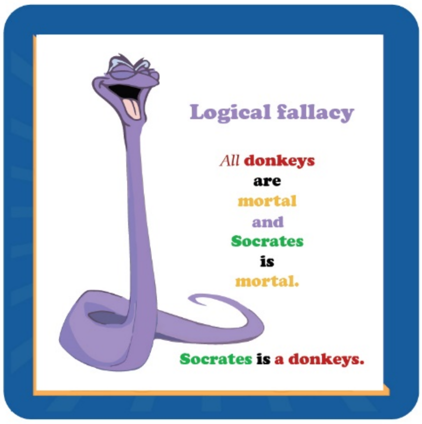
Hence the title of Game 7: The Robot. At first glance, this pop-up has no other use than to allow children to visualise the Truth Tables, without having to learn them by heart, such as a cheat sheet. It is my first interpretation of Wittgenstein's proposal (1922, point 5.43): 'But in fact all the propositions of logic say the same thing, to wit nothing[130]. Emptied of all substance and reduced to pure form, logical propositions are nothing but tautologies such as multiplication tables. However, bridging the gap between traditional logic and 'Calculus logic' reinforces here Robert Blanché's point of view. As Blanché (1970, pp. 350–353) points out, the adepts of mathematical logic thought they could dispense with the axioms and classical rules of logic by stating in the very language of calculation the procedure to be followed to obtain a conclusion, i.e. without having to refer to the axioms and rules of the detachment that make it possible to move from the premises to the conclusion. In other words, it would no longer be necessary to refer to Aristotle's *dictum de omni et nullo*, nor to the rule of the *modus ponens* of the Stoics. Nevertheless as one can indeed see in Games 7.2 and 7.3, this logic, without metalanguage, that Wittgenstein's Truth Tables innovate, uses material implication (noted =>) to move from the premises to the conclusion instead of the 'therefore' that Lewis Carroll and Russell had so many problems with[131]. This is a fact, the metalanguage has not disappeared. To move from material implication to conclusion, a rule of detachment must be used in the *modus ponens*: if p implies q, then q can be detached, provided *only* that p is asserted. Other metalinguistic principles that have not disappeared in Boolean Truth Tables is the principle of non-contradiction and the principle of the excluded third party (0 or 1). In Truth Tables, contradiction is a tautology noted 0 and truth is a tautology noted 1. Moreover, in a conversation, in a discourse, one does not always have at his/her disposal a computer or a game board to calculate the Truth Tables. It may be useful to keep in mind the rules of logic that make reasoning valid or invalid. After using game 7, players will be able to return to the puzzles of games 1 to 3, using the timeless rules of logic, as shown in the following example.

[130] Wittgenstein L., 1922. *Tractatus Logico-Philosophicus*. Reprint 2014. Translated from German by D.F. Pears and B. F. McGuinness. Introduction by B. Russell. London and New York: Routledge Great Minds, p. 53.

[131] This way of characterising what today is called truth functions, which include what is called 'material implication, noted =>, was known from the Megarian school of philosophy, known as the Dialectical school, of which three names have come down to us: Eubulide is known for the Liar's Paradox, Diodorus and Philo of Megara, the Dialectician (Blanché, 1970, p. 99). Philo's conception of implication corresponds in the current symbolism to Russell's material implication. This connector (=>) has given rise to paradoxes such as Lewis Carroll's paradoxical puzzles: *What the Tortoise Said to Achilles* and *The Barber Shop Paradox* to which Russell refers to his *Principles of Mathematics*. Theses paradoxes come from the confusion between the notion of implication and the notion of inference or deduction, i.e. between the language of logic (the connector used in the truth table as a means of calculation) and the metalanguage of logic (the connector used as a criterion for judging the validity of deductive reasoning announced by the term "therefore" in a syllogism (Gattégno, 1966, p. 270). As Bartley (1977, Appendix C, Editors' note, p. 468) writes: 'The validity of an argument, as opposed to the truth of the conclusion, must be defended metalinguistically."

QUIZ

PUZZLES AND LOGICAL RULES
TO CHECK THE VALIDITY OF A REASONING



The syllogism is in the form of the second figure *AAA***2**:

All **P is M**
All **S is M**
All **S is P ?**

Premises	All P is M	All S is M	All S is P	Premises	All P is M	All S is M	All S is P
Rule 1. ***Terminus esto triplex*** Three propositions , three terms and a middle terme	✓	✓	✓	**Rule 5.** ***Ambae affirmantes*** If premises affirm conclusion cannot be denied.	✓	✓	✓
Rule 2. ***Latius hos*** Terms: no more extension to the conclusion than in premises			✓	**Rule 6.** ***Utraque si praemissa*** No conclusion If two negative premises.			
Rule 3. ***Nequaquam medium*** No middle term in the conclusion			✓	**Rule 7.** ***Pejorem sequitur*** If premises negative, the conclusion is negative. If particular, the conclusion is particular.			
Rule 4. ***Aut semel*** Middle term must be universal in at least one of the premises			✗	**Rule 8.** ***Nihil sequitur*** No conclusion If two particular premises.			

Symbols:

✓ True ✗ Fault

Undistributed middle (Rule 4): As the syllogism of Ionesco's Logician, the conclusion of this syllogism is invalid.

To conclude on the results obtained

It is more than 80 concepts[132] of formal, symbolic and abstract logic that are illustrated here through visual arts: pop-ups, game boards, playing cards, counters and figurines. It should be noted that the objective was to see at what level of detail the visual arts could contribute to illustrating abstract concepts and complex reasoning. The nine illustrated Booklets allow children to start playing quickly (with the 'quick start' section) and consider the level of detail that suits them. Thinking of older children interested in logic who would like to delve deeper into the issues illustrated here, special attention has been given to the Booklets, the bibliography and tutorials referenced on the internet to enable them to go even further.

Overall conclusion and perspectives

In the final part of this dissertation, I outline a general conclusion (Chapter VIII) and reflections, suggestions for research, perspectives and innovative applications (Chapter IX).

Chapter VIII. Summing up

8.1 Opening remarks

Having reached the end of my research, it is time to examine the following two questions: What have I learned? What could other researchers acquire from this?

To provide answers, I return to the key issues presented in the introduction and the preliminary chapter. These were questions I was not quite certain I could answer at the time. I set myself nine categories of criteria (part I, ch.3) to carry out this research. The most important thing, given the topic, was to understand what the models of pure deductive sciences are (criterion 1), and to be able to illustrate their fundamental principles. It is not for me to judge the aestheticism of my drawings and prototypes (criterion 8). On the other hand, as attested by the nine Booklets that accompany each game, I made a point of valuing the first criterion. This consists of trying to comprehend the text, namely the abstract ideas and concepts to be illustrated, that are well understood by professionals, but very little taught at school.

[132] In his glossary/index, Hurley (2005, ed. 2008, pp. 672–682) defines almost 300 terms related to formal, deductive, inductive and informal logic. In his glossary devoted, as here, to formal deductive logic, Lee (2017, pp. 305–315) defines nearly 140 terms including the logic of predicates not dealt with here. Approximately 100 definitions are of direct relevance to the topics covered here. Comparing the two lists, and referring to Lee's, I can conclude that more than 80 fundamental abstract and complex concepts have been illustrated through the visual arts: conversion, obversion, transposition, conjunction, disjunction, *modus ponens*, *modus tollens*, paradox, dilemma, etc. Concerning the modes of reasoning of standard logic, the illustrations here cover the two main fields of formal deductive logic: categorical logic and propositional logic.

Pure sciences as symbolic and formal logic are fundamentally based on axioms and postulates, most of the time non-demonstrable. These are the strange concepts that Euclid demanded to be accepted before any demonstration. They do not have the constraint of the experimental sciences necessitating actual experimentation. In the same vein, the questions stated at the beginning of the research were not hypotheses whose validity was to be empirically tested, but postulates. Their purpose was to determine whether it was possible to deduce anything of interest for research in illustration.
As Adrian Wallwork recommends (2011, p. 269), I have limited my conclusion (point 8.2) to fewer than 250 words and to the following five points. As I do not have any experimental tests to discuss that bring nuances to the conclusion, I will discuss afterwards (chapter XIX) some thoughts on these five points.

8.2 Conclusion

I can sum up my research by means of two concepts that I have used and which I define as follows[133]: in practice, the 'Pop up Game' allows me to illustrate complex and abstract reasoning and, in theory, what I call 'Visual Thinking' is the process that allows me to illustrate in the field of formal logic the metalanguage of the Art of Thinking or Critical Thinking. The production of nine prototypes and the associated illustrated Booklets gives an affirmative answer to the following five questions asked beforehand.

TO SUM UP

1. It is possible to illustrate through visual art, abstract and complex concepts and reasoning as they exist in pure sciences

2. The creative visual art is an appropriate medium for establishing a visual link between art and science.

3. It is also possible to teach abstract concepts and reasoning to children through the creative visual art and to illustrate what seems to be un-illustratable.

4. A creative visual link can be established between rationality and fantasy.

5. If it is difficult to affirm here that new concepts and theories can be discovered through visual thinking, because I merely illustrated concepts and reasoning well-known to the specialists, at least it has been possible to reinterpret the Carrollian concept of nonsense in an illustrative and logical way.

[133] These definitions do not refer to those of authors or game publishers who sometimes use the same terms: Pop-up games and Visual Thinking in another context than the one studied here.

Chapter XIX

Reflections, suggestions for research, perspectives and new applications

9.1 Reflections on five conclusions

1. The first point of my conclusion is an application of a logical principle. It cannot be said that 'it is impossible to illustrate reasoning that makes it possible to progress from axiomatic concepts to conclusions using rules of inference and validation', if there is at least one example that shows this is possible. This principle is illustrated by the diagonal of the Square of Opposition (Game 4). If 'Some S is P' is true, then 'No S is P' is false. This example (some S is P or equivalently some Prototypes P are examples S), which I have been seeking for a long time has now been materialised by using the nine prototypes I have made. Other illustrators may discover other ways than my prototypes to illustrate these concepts, but this is the strength of logic, it will only reinforce the conclusion. On the other hand, one cannot conclude from a few examples that all abstract concepts and reasoning can be illustrated. The learning of logic through the Booklets shows that the illustrations complement the explanations provided by the text. As the case studies have shown (part I, ch. 2.6), this raises the crucial question of the image/text ratio where illustrations supplant the text in a way that is erroneous or contrary to the intentions of the author of the texts.

2. As Leonardo da Vinci had already shown with his prototypes, creative visual art is an appropriate medium for establishing a visual link between art and science. Without comprehending the concepts of 'practice-based research' and 'practice-led-research', Leonardo da Vinci had conceived a method that became current in science. This can be applied in illustration. It consists of three steps, to be carried out in order. First the practice, then the theory, finally the practice. It is this method that I finally rediscovered and followed.
This method could be summarised in a unique formula: 'ideology + ideography'.
The term 'ideology' is used here in the sense defined by Destutt de Tracy (1754–1836) in his famous *Memory of the Faculty of Thought* (1798), which is now called *Project of Elements of Ideology* (1801)[134]. Ideology is an 'operation of the mind, which consists of gathering several ideas into one, to which is given a name that unites them'. This noun can be 'concrete' for adjectives such as pure, good, great, etc., which express a quality unit to a subject. In comparison, we can give the name 'abstract' for terms such as purity, goodness, greatness, etc. They express quality separated from any subject. This sums up one of the significant problems of illustration. In practice, if it is not challenging to draw someone tall, short, bald or hairy, it becomes more complex to draw his or her particularities that make him or her admirable or otherwise.

[134] Destutt de Tracy, A.L.C. 1801. *Projets d'Eléments d'idéologie*. Reprint 2004. Paris: L'Harmattan, chapter VI, De la Formation de nos Idées composées ("The Formation of Our Composite Ideas"), quotation p. 82, whose I translate the title of the book and chapter into English.

The second term 'ideography' is used by the father of modern logic, Gottlob Frege (1848–1925) in his *Ideography* (1879). It is a fully formalised language invented by the logician, made up of signs and symbols. It gave me the idea to create some symbols and to use visual colour codes to represent signs and logical operations. This is the beginning of the creation of a formal syntax for the language of illustration. My ideography is made up of playing cards, figurines, counters, pieces of wood to move and images assembled like puzzles. These artifacts represent axioms and rules of reasoning. They serve to fight fallacious reasoning and thwart the traps of paradoxes. The language of logic illustrated here has the magical and strange power to help us reason correctly and to overcome syllogistic argument and fallacious rhetoric. Here again, other illustrators may, if they wish, create their ideography to illustrate the power of symbolism that exists in the pure sciences and many other fields.

3. As Lewis Carroll has shown in many of his well and lesser-known books, that it is possible to establish a link between storytelling and logic. Since the aim of this thesis is to translate abstract concepts and reasoning into creative visual arts, I have found nothing better than to use the concept of strategy and tactical games, associating the idea of play with a pop-up. What I call 'pop-up games' is for the purpose of illustrating the Venn and Lewis Carroll's diagrams and Wittgenstein and Boole's Truth Tables. In addition to the rules of the game, I use pop-ups to pedagogically illustrate the rules of logic, in an entertaining way. However, as Lewis Carroll pointed out in his introduction to *Symbolic Logic*, logic games and pop-ups cannot dispense with a thorough study of logic. This is the purpose of the illustrated instruction manuals (Booklets) associated with the games.

4. Through these games and prototypes, we can see theories are made up of concepts and rules, deconstructed with other concepts and rules and reconstructed with new concepts and rules (when they are not old concepts and rules). It is this game of combinations that makes it possible to indefinitely combine those concepts, as seen in Georges Perec's *Life: A User's Manua*l (1978) or in Jorge Luis Borges' *Fictions* (1941). These two authors have succeeded in associating the combinatorial analysis of mathematics with a work of pure fiction. This makes it possible to establish a link between rationality and fantasy. Other illustrators will be able to find many other examples.

5. This research has offered me the opportunity to deconstruct, reconstruct and decompartmentalise several disciplines of knowledge, particularly in the field of pure sciences and the arts. Thanks to Lewis Carroll and his logical tales, I was able to establish a link between logic and tales. The deconstruction and reconstruction of the Carrollian puzzle especially, allowed me to discover that there was in this mathematician and logician another possible reading of his work.

The Carrollian nonsense is ordinarily understood as a simple entertainment. Yet, underlying this, there are more abstract paths behind nonsense that give access to pure knowledge and reasoning. To discover them, another means of interpretation was necessary, that of logic. A path that I have discovered through illustration and that might inspire other researchers.

9.2 Suggestions for research, perspectives and new applications

My research was limited to logic and pure sciences, that is, deductive reasoning models. Like syllogisms, they are models that start from axioms and postulates and proceed to a certain conclusion through rules of inference and validation. Other researchers may choose to illustrate other reasoning models. For example, the inductive and analogical model of experimental sciences or in the field of logic where it is known as logical empiricism. As Sophie Rabau and Florian Pennanech (2016) have shown[135], we can also create literary theories and therefore propose to illustrate their metalanguage. This goes far beyond the strict domain of pure and experimental sciences.

According to my approach, if I were to illustrate the History of Art with the same rigour as this study, I would start by trying to understand (criteria category N°1) the metalanguage used in each era by artists, i.e. the axioms and rules used. This is what Anne d'Alleva (2004) calls 'approaching theories of art-historical practice[136]'. I would then use the ideography of my chromatic circle inscribed in an enneagon defining 9 categories of criteria, as well as the forms, symbols and signs contained in the 9 pop up games. With this approach, Leonardo da Vinci's Golden ratio and the Fibonacci sequence would not be considered an analogy with the beauties of Nature, but as an axiom defining an aesthetic criterion that some artists use, unlike others. Alberti's Laws of Perspective (*On Painting*, 1435) – which after all are trompe l'oeil – would be classified among the rules of the metalanguage of Art. Based on axioms and the metalinguistic rules, I would try to see what can be concluded from the works studied. In the end, it is this knowledge that one may wish to learn and master that I will try to transmit employing visual art, here called Visual Thinking or Thinking Art.

[135] Rabau and Pennanech, 2016. *Exercices de Théorie littéraire* (I translate into English: "Exercises in Literary Theory"). Paris: La Sorbonne Nouvelle.

[136] Alleva, A.d'., 2004. *Methods & theories of Art History*. Reprint 2019, 2nd edition. London: Laurence King publishing, p. 1 and 'The analysis of form, symbol, sign, iconography, semiotics, systems and codes, art puzzles, word and image, etc. pp. 16-44.

In the History of Art, for example, if I had to illustrate the abstract concept of Marcel Duchamp's readymade (1916), as the Art of Logic recommends, I would go deeper into the axioms or definitions it contains, for example: 'An object is a work of art by the simple choice of the artist[137]' or I could consider the opposite: It is the spectator who, by looking at it, creates the work of art, or I could postulate that 'A work of art is what is exhibited in a museum', and because Marcel Duchamp's porcelain urinal (Fountain, signed 'R. Mutt', 1917) is exhibited in museums[138], one could deduce that it is a work of art[139]. Of course, other definitions and conclusions could be discussed. However, the interest in using the model of deductive, abstract and formal logic and thought experiments is to be able to imagine objects (axioms, postulates, ideas) which do not yet exist, or which are not visible or accessible to our senses. With the evolution of science, art and ideas, it is likely it will no longer be possible to solely illustrate the unknown by the known. How to illustrate the non-Euclidean abstract geometries of modern physics, or the new quantum physics where neither particles nor waves, in the classical sense of the term, exist? How to illustrate the invisible matter in the n-dimensional space-time that the mind conceives by reason, but that we cannot see?

Suppose for a moment that the famous science fiction writer Orson Scott Card (2018)[140] is right when he considers that since the end of the Victorian era by attempting to illustrate the unknown by the known: 'Science has stopped providing us with material for our stories.' Science! —What happened to dreams, the wonderful, the fiction? Robots no longer make us dream; they have become a reality. A science that is not interested in fiction, dreams and tales, which forgets in its equations, feelings and emotions, is presumably a science doomed to die. This may be the message of 'scientific' tales. At least this question of the relationship between science and storytelling offers alternative avenues of research for illustration.

To conclude, the thesis could be useful to students studying in the field of children's picture books in three complementary areas: the illustration of abstract and complex concepts and reasoning as in the pure sciences, the prospect of further developing the use of illustration in disciplines that share some of the basic abstract and complex concepts discussed in this thesis, and exploring the possibility of an educational art movement ('The Art of Thinking') that has yet to be better qualified, on which I, with other students, could continue research beyond this thesis. In practice, my prototypes have been developed to illustrate over 80 concepts of formal, symbolic and abstract logic, some of which are common to several disciplines. As the artefacts can be adapted by other image makers, the large number of prototypes can provide an illustrative playground for other students researching children's picture books or related fields.

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[137] André Breton (1938). *Dictionnaire abrégé du Surréalisme*. Paris : Galerie des Beaux-Arts : He defined readymades as "manufactured objects raised to the dignity of works of art through the choice of the artist'.

[138] For example: London, Royal Academy of Arts, *Dali/Duchamp*. The exhibition, 7 October 2017- 3 January 2018.

[139] If one accepts this definition of a work of art, the conclusion is necessarily valid. It is a syllogism of the standard form Barbara AAA1:
All object exhibited in a museum (M) is a work of art (P).
Marcel Duchamp's porcelain urinal (S) is an object exhibited in a museum (M).
Therefore, Marcel Duchamp's porcelain urinal (S) is a work of art (P).

[140] Card, O. S., 2018. Les chefs-d'oeuvre de la science fiction. Paris : *Le Point Pop, N° 25, Hors-série*, Octobre-Novembre, pp. 91-93, remarks collected by Phalène de La Valette. I translate the citation into English.

Appendices to Chapter VII, Part III

Contents

Appendices

7.2.1 Using visual arts to visualise the Art of definition, classification and concept

The chromatic circle inscribed in an Enneagone allows to visually define and classify axioms, principles and rules of a theory according to the objectives set (here nine categories of criteria). This tool is employed here to represent the principles and rules of standard logic.

TABLE 85

A CHROMATIC METHOD FOR A STEP-BY-STEP INTERPRETATION OF A DEDUCTIVE THINKING IN TERMS OF AXIOMS, RULES AND RESULTS

The colour wheel.

- The outer circle contains the 9 categories of criteria proposed, divided into three groups: Group 1 (dark red, pink, ochre), Group 2 (three different shades of yellow) and Group 3 (three different shades of blue-green).

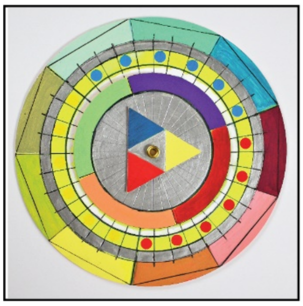
- The inner circle is divided into four equal parts, intended to validate the general rules (2a in orange), the reasoning (2b.1, in green), the results (2b.2 in purple) and the application of the rules of the game (2b.3 in red).

- In the central equilateral triangle, the colours facilitate the mental representation of the three main concepts of deductive thinking: the axioms, in red, the rules of inference and validation, in yellow, and the results, in blue. By rotating the central triangle, these concepts can be associated in turn with the 9 categories of criteria (or objectives) written on the enneagon (1. objective of understanding the text to be illustrated, in red, 2. objective of efficiency of the image/text ratio, in pink, etc.).

Example of a three-step application of Aristotelian syllogisms

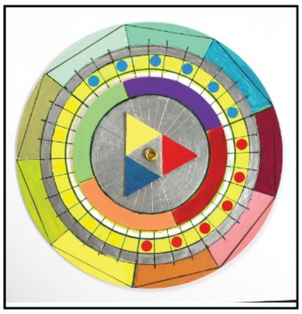
Step 1. Understanding analysis of the three main principles of Aristotelian logic: the principle of identity, the excluded third party and the principle of non-contradiction.

In the red triangle, these principles are associated with criterion 1 of understanding, in dark red, on the outer circle.



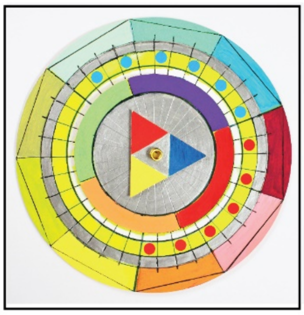
Step 2. Understanding analysis of the fundamental rules of Aristotelian logic: the principle of the *dictum of omni et nullo* and the middle term

The central triangle is positioned on criterion 1 (understanding) to examine the rules of syllogisms, the yellow triangle. In this example, the inner circle is rotated here to examine the rules for validating syllogistic reasoning (*the dictum of omni et nullo*), in green. From Latin: 'the maxim of all and none', it postulates the following: whatever is affirmed or denied of an entire class or kind may be affirmed or denied of any part of it. This law gives an important role to a middle term that allows a valid conclusion to be inferred from the stated arguments (or premises).



Step 3. Understanding analysis of the fundamental rules of validity of the conclusion of an Aristotelian syllogism.

The central triangle is positioned on criterion 1 (understanding) to examine the validity of the results, in blue. In categorical syllogisms, the principal rule is that a syllogism is valid if and only if its conclusion follows from its premises.



In the same way, by rotating the central triangle and the inner circle, these three steps can be applied to the other categories of criteria inscribed on the Enneagon.

As the bibliographical references indicate, according to the authors the logic can take different names: Logic, Art of Thinking, Art of Argumentation, Critical Thinking, Visual Thinking. The term 'logic' is then defined as an extension of the basic Aristotelian and Stoic models. In addition, as the History of logic shows[141], logicians have used several names and visual methods to classify and enumerate things in an exhaustive, geometrical and combinatorial way: circles in Euler's and Venn's diagrams, squares in Lewis Carroll's quadriliteral diagram, Square of opposition, cross-tables in Wittgenstein's Truth Tables, heuristic schemes in Tony Buzan's mind mapping, tree method in Porphyry's tree[142], Platonic division (or dichotomy, A, non-A, B, non-B), etc. Here, a question of definition arose: what can the process of using the visual arts to illustrate the art of thinking be called? In conclusion of my research and in a purely arbitrary manner, I call this process 'Thinking Art' according to the following formula:

$$\boxed{\text{Thinking Art} = \text{Visual Art} + \text{Art of Thinking}}$$

The contraction of the two words or the disappearance of the middle term 'Art' makes it possible to rediscover what some authors call Visual Thinking.

Aristotle's reasoning model

Let us begin by summarising the basic model of Aristotle used by Euler, Venn and Lewis Carroll. In this model, the principle of exhaustive enumeration of all possible cases plays a crucial role in the reasoning process. This can be represented visually by means of a cross table. Before doing so, it is necessary to briefly recall what the basic model is, knowing that it is detailed in Booklets 1 to 3 in particular.

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[141] Moktefi, A. and, S.-J. Shin (Eds.), collective work, 2013. *Visual reasoning with diagrams*. London: Springer Basel, Birkhäuser. Arnheim, R., 1969. *Visual thinking*. Reprint 1997. Berkley and Los Angeles, California, London: University of California Press, Ltd. Turetzky, P., 2019. *The elements of arguments. An introduction to critical thinking and logic*. Peterborough, Ontario: Broadview Press. Arnauld, A. and P. Nicole, 1662. Reprint 1993. *La Logique ou l'Art de penser*. Paris: J. Vrin.

[142] Zabarella, I. (1533–1589), translated from Latin into French by M. Bastit, 2003. The process of Dichotomy using *Tree of Porphyry* is translated into English and illustrated in Sidgwick, ed. 2015, pp. 113–114): Substance: corporal, incorporal; Corporal, body animated, body inanimated… Animal: rational, irrational; Rational, Man: Socrates, Plato, Aristotle and others.

TABLE 86
PERSONAL ILLUSTRATIONS

STANDARD FORM, MOODS AND FIGURES OF ARISTOTELIAN REASONING

To sum up, the Aristotelian reasoning consists of two premises and a conclusion. Each premise consists of a subject (S), a verb and a predicate (P). The two premises have one term in common: the middle term (M) that allows the conclusion to be deduced. The four positions of the middle term (M) in the premises define four different figures called Figure. I, II, III, IV.

Figure I

M – Predicate
Subject – M

Subject – Predicate

Figure II

Predicate – M
Subject – M

Subject – Predicate

Figure III

M – Predicate
M - Subject

Subject – Predicate

Figure IV

Predicate – M
M - Subject

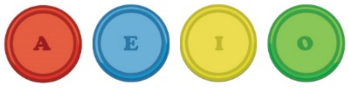
Subject – Predicate

Each of these three propositions are defined according to two criteria: quality (affirmative or negative proposition) and quantity (universal or particular proposition). The four forms of propositions are coded A, E, I, O; from Latin: A and I as in *affirmo*, E and O as in *nego*, that can be found visually in the Square of Opposition.

A means Universal and Affirmative proposition; E: Universal and Negative proposition; I: Particular and Affirmative proposition; O: Particular and Negative proposition. These logical propositions are symbolised in the games by coloured pieces.

TABLE 87

EXAMPLE OF A VALID MODEL OF REASONING IN FIGURE I, KNOWN AS BARBARA[143]



AAA1 Barbara

All **animals that drink milk when they are babies are mammals;**
All **people are animals that drink milk when they are babies.**

All **people are mammals.**

Or in its standard form:

AAA1 Barbara

All **M are P;**
All **S are M.**

All **S are P.**

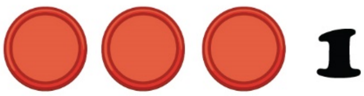
Symbolic form[55] , Figure I

$\mathbf{A}_{MP}$
$\mathbf{A}_{SM}$
$\mathbf{A}_{SP}$

By applying the symbol $\models$ used in the calculation of propositions to designate a valid conclusion here, the logical formula can be written on one line:

$\mathbf{A}_{MP}, \mathbf{A}_{SM} \models \mathbf{A}_{SP}$

This form of **AAA1** propositions is represented in the games by the following counters (from left to right, 2 premises: $\mathbf{A}_{MP}$, $\mathbf{A}_{SM}$, 1 conclusion: $\mathbf{A}_{SP}$ and Figure I):



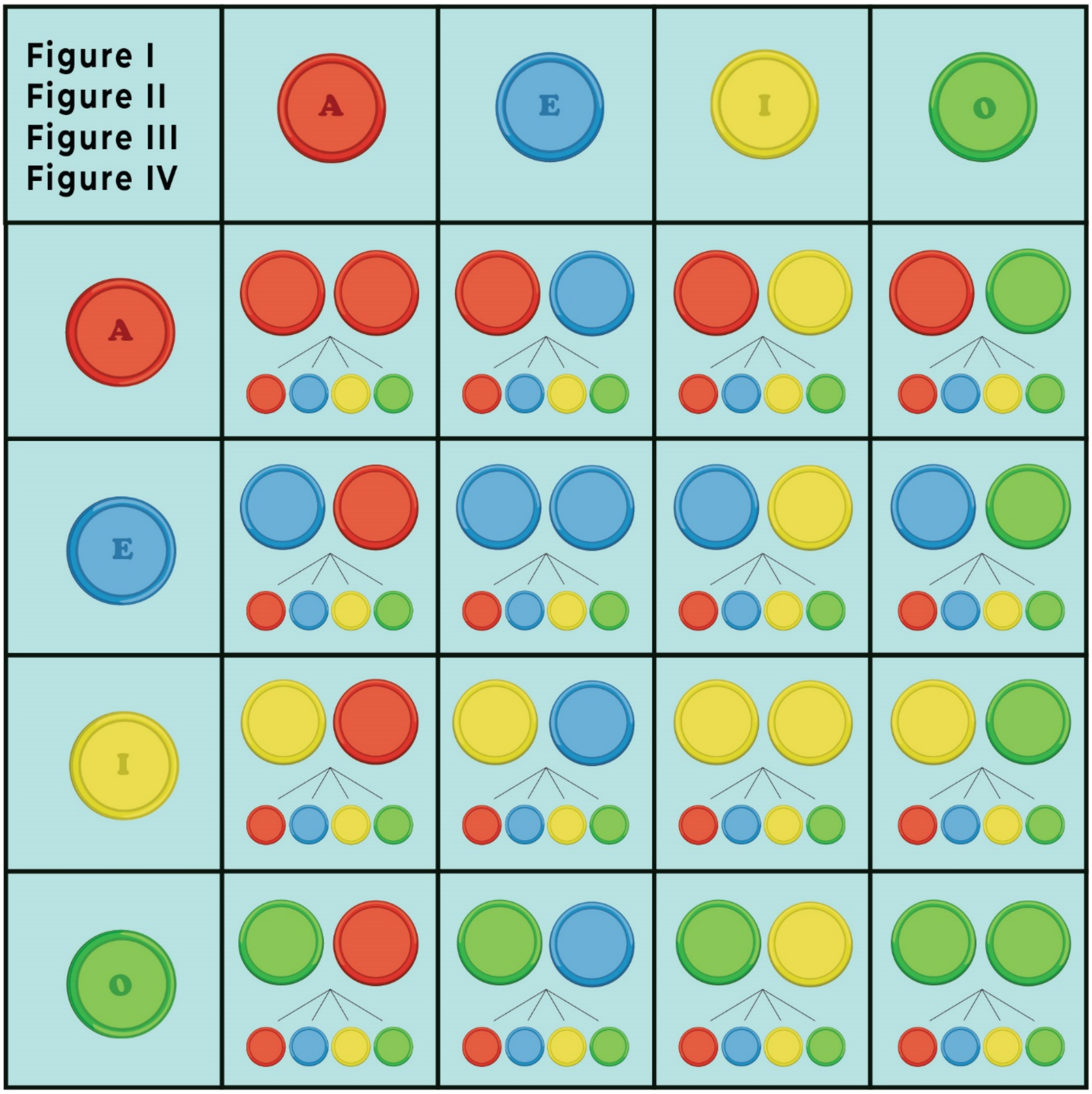
143 This presentation is proposed by Ruggero Pagnan, *A Diagrammatic Calculus of Syllogisms,* pp. 33–53 in Moktefi, A. and, S.-J. Shin (Eds.), collective work, 2013.

The principle of exhaustive enumeration takes into account both the order of the letters A, E, I, O and the possibility of repeating them in different syllogisms and figures. The two premises of the EA form do not provide the same conclusion as the two premises of the AE form. In the language of combinatorial calculus, the letters are taken 'successively' (orderly), with possible repetition. Using the cross-table and tree methods, this gives 16 combinations of 2 premises x 4 Figures x 4 possible conclusions = 256 possible syllogisms, valid and invalid.

TABLE 88

EXHAUSTIVE VISUAL ENUMERATION OF THE 256 POSSIBLE MODELS OF ARISTOTELIAN CATEGORICAL REASONING

This cross-table shows the pairs of possible premises indicating the possible valid and invalid conclusions to which they lead.



Once this first step has been completed, it remains to visually determine which syllogisms are valid. This requires the definition of judgement criteria.

7.2.2 Using visual arts to visualise the Art of judgement

After the first step of the exhaustive enumeration, the second step is to highlight three basic logical concepts of good arguments: truth, validity and soundness. In order to select valid reasoning models, a distinction has to be made between truth and validity criteria. A conclusion will be stated to be valid (or invalid) depending on whether or not it complies with certain rules of deduction. As in the fairy tales: ‘Once upon a time in a faraway land’, the Aristotelian logic can be timeless. This logic can go so far as to disregard the meaning and reality of propositions and retain only their *possibility* of truth or falsity. Lewis Carroll frequently exploits this possibility in children’s tales, such as the syllogism mentioned in the Booklet of Game 5 where what really counts is the validity of the reasoning.

TABLE 89

LOGIC AND FAIRY TALE

Lewis Carroll’s Syllogism:

All cats understand French.
Some chickens are cats.
Conclusion: some chickens understand French.

Here is his comment in an amusing way:
‘Also the three propositions are so related that, if the first two were true, the third would be true. (The first two are, as it happens, not strictly true in our planet. But there is nothing to hinder them from being true in some other planet, say Mars or Jupiter – in which case the third would also be true in that planet, and its inhabitants would probably engage chickens as nursery governesses. They would thus secure a singular contingent privilege, unknown in England, namely, that they would be able, at any time when provisions ran short, to utilise the nursery governess for the nursery dinner!)’
(*Symbolic Logic*, Reprint 2015, New York: Dover Publications, text pp. 57–58 and diagram pp. 61–62)

In this Carrollian syllogism: ‘Some chickens understand French’ is a valid conclusion, whereas in the reality of our world that conclusion is not necessarily true. Displayed and illustrated in game 7.1, players will have the opportunity to learn that a valid reasoning may have the following combinations:

- true premises and true conclusion
- false premises and true conclusion
- false premises and false conclusion.

However, a valid reasoning can never have:

- true premises and false conclusion.

These definitions make it possible to establish a link between science and fiction, insofar as the premises do not need to be related to known reality in order for a reasoning to be considered valid. Nevertheless, an argument is said ‘a sound argument’ if it is a valid argument and the premises are all true (in *our* planet or in some *other* planet).
To sum up: From this theoretical and practical approach, I deduce a main conclusion. The Aristotelian theory of categorical syllogism corresponds to three essential principles, which I interpret and illustrate in the games as follows:

TABLE 90

THE THREE PRINCIPLES OF THE ARISTOTELIAN MODEL OF REASONING

- The first principle is to consider all possible and imaginable cases (4 figures, 4 forms of proposition A, E, I, O, two premises and one conclusion). It is the ‘principle of exhaustiveness’. It is based on the theory of combinatorial calculus.

- The second principle consists of establishing rules which make it possible, on the basis of intuitive, obvious or a priori defined criteria or axioms, to eliminate incorrect reasoning and to retain only the valid conclusions.

- The third principle is that these rules and criteria cannot be purely arbitrary. They must lead to results deemed useful and effective at least by those who established the axioms and rules.

In the *Organon*, Aristotle formulated for categorical syllogisms rules or characteristics common to valid deductive reasoning. Tradition holds eight rules and corollaries[144]. The first four concerns the terms of the syllogism, the last four its premises and conclusion. These fundamental rules, which are part of the metalanguage of logic, are partly found in the Square of Opposition (Game 4) and detailed in Booklet 7.1. They allow to select valid syllogisms and conclusions.

Here are the rules of the categorical syllogism[145]:

[144] There are several translations and presentation of these rules which according to the authors can be introduced in a different order. For example, Arnauld and Nicole (1662, ed. J. Vrin, 1993, pp. 182–188 and Latin versification, notes 258 to 263) presents 6 rules and 6 corollaries; Hurley (2008, p. 283), presents 5 rules; Kreeft (2014, p. 243) presents 6 rules and 2 corollaries.
We will retain here the order of mnemonic verses which is also the order chosen in particular by Tricot (1973, pp. 201–204) and Thibaudeau (2006, pp. 731–732). The Latin verses are indicated in Booklet 7.1.
[145] These rules are summarised in very ancient mnemonic verses which were found in a *Synopsis of the logic* of Aristotle by Michel Psellus dating from the 11th century (Tricot, ed. 1973, pp. 201–204 and Chenique, pp. 209–213, and p. 209, footnote 2).

TABLE 91

RULES FOR VALIDATING ARISTOTELIAN CATEGORICAL SYLLOGISMS

1. Four general rules concern the terms of categorical syllogisms:

Rule 1. **Terminus esto triplex**: medius, majorque, minorque[146].
A syllogism contains three propositions, and no more: major, minor and middle term.
If there are fewer than three terms, it is an immediate inference.
If there are more, it is a poly syllogism, called sorite.

Rule 2. **Latius hos quam praemissae conclusion non vult.**
The terms should not have more extension in the conclusion than in the premises.
No term must be distributed (universel) in the conclusion unless it is distributed (universel) in the premises.

Rule 3. **Nequaquam medium capiat conclusio opertet.**
The middle term should not be into the conclusion of which it is the generator.

Rule 4. **Aut semel, aut iterum medius generaliter esto.**
The middle term must be universal in at least one of the premises.

2. Four general rules concern the propositions of categorical syllogisms (premises and conclusion):

Rule 5. **Ambae affirmantes nequeunt generare negantem.**
If the premises affirm, the conclusion cannot be denied.
A negative conclusion cannot be proven from two affirmative propositions.

Rule 6. **Utraque si praemissa neget nil inde sequitur.**
From two negative premises, nothing can be concluded.

Rule 7. **Pejorem sequitur semper conclusio partem.**
The conclusion should not be stronger than the premises.
If one of the premises is negative, the conclusion is negative.
If one of the premises is particular, the conclusion is particular.

Rule 8. **Nihil sequitur geminis ex particularibus unquam.**
From two particular premises, nothing can be concluded.

3. Specific rules[147] to the 4 Figures I, II, III, IV.
There are specific rules for each Figure to quickly eliminate invalid syllogisms (detailed in Booklet 7.1).
For example, specific rules of Figure I:
Specific rule 1. The major premise must be universal (A, E).
Specific rule 2. The minor premise must be affirmative (is, are).

If these rules are not respected, the syllogism is invalid.

Deduction related to Figure I
a) IA, II, OA, OI combinations do not comply with Rule 1. They must be eliminated.
b) AE, AO, EE, EO, IE, IO, OE, OO combinations do not comply with Rule 2. They must be eliminated.

Consequently, only the forms AA, EA, AI, EI remain valid. Tradition[148] has called these valid syllogisms: Barbara (AAA1), Celarent (EAE1), Darii (AII1), Ferio (EIO1).
The other valid Figures can be converted into Figure I (Games 2 and 3).

7.2.3 Using visual arts to visualise the Art of demonstrations: proof and calculation

[146] The Latin verses are indicated here and in the booklet 7.1 because sometimes logicians designate these rules by their first Latin words, and say for example that such an incorrect syllogism is a ‘Latius hos’ (Chenique, p. 209, footnote 2).
[147] The following rules allow a quick determination of whether a conclusion is valid or not: Arnauld and Nicole, ed. 1993, pp. 191–203, Sidgwick, ed. 2015, p. 34, Tricot, ed. 1973, pp. 197–203.
[148] These mnemonic names, such as Barbara, Celarent, etc. are often attributed to Petrus Hispanus (1210/1220-1277), a professor in Siena around 1246, concentrated on medicine, theology, logic, physics, metaphysics, and Aristotle’s dialectic, who became pope in 1276 under the name of John XXI and died in 1277. These mnemonic names are said to have been inserted in the 14th century in translations of ancient Greek texts (Arnauld and Nicole, ed. J. Vrin, 1993, note 269).

The third and final step is to determine which of the 256 possible reasoning models are valid. This can be done in several ways, first by using the puzzle method (Games 1 to 3), then by applying the visual diagrams of Venn and Lewis Carroll (Games 5 and 6), or by directly following the 8 rules and corollaries of Aristotelian logic (Game 7). Among the exhaustive enumeration of all possible categorical syllogisms, only 15 or 24 models of reasoning are valid (depending on the hypotheses made[149]). Theses models are historically classified in Figures I to IV, which tradition has given each valid syllogism a name: Barbara, Celarent, Darii, etc. These figures are illustrated in the games on playing cards or as figurines to be placed on a game board. Examples of syllogisms are given in the Booklets for each of the following names.

[149] According to their assumptions, Lee (2017, p. 169) retains 'only 15 valid argument forms of categorical syllogism ', as does Kaye (2009, pp. 46–50).

TABLE 92

SUMMARIES OF ILLUSTRATIONS OF 24 VALID SYLLOGISMS

Figure 1 and code names
AAA1 Barbara, EAE1 Celarent, AII1 Darii, EIO1 Ferio and AAI1 Babari, EAO1 Celaront.

Figure 2
AEE2 Camestres, EAE2 Cesare, EIO2 Festino, AOO2 Baroco, and AEO2 Camestros, EAO2 Cesaro.

Figure 3
AAI3 Darapti, AII3 Datisi, IAI3 Disamis, EAO3 Felapton, and EIO3 Ferison, OAO3 Bocardo.

Figure 4
AAI4 Bamalip, AEE4 Calemes, IAI4 Dimatis, EAO4 Fesapo, EIO4 Fresison, AEO4 Calemos.

The fourth figure has had several statements in the past, but Aristotle does not take it into account, seeing it as a pure language game. The following cross-table shows 24 valid reasoning models (15 from Fig. I to III, 19 from Fig I to IV and 5 with the hypothesis of 'existential import', a rule used to assume the existence of subject or predicate elements). Players will be able to check with the 8 rules that the following syllogisms are valid.

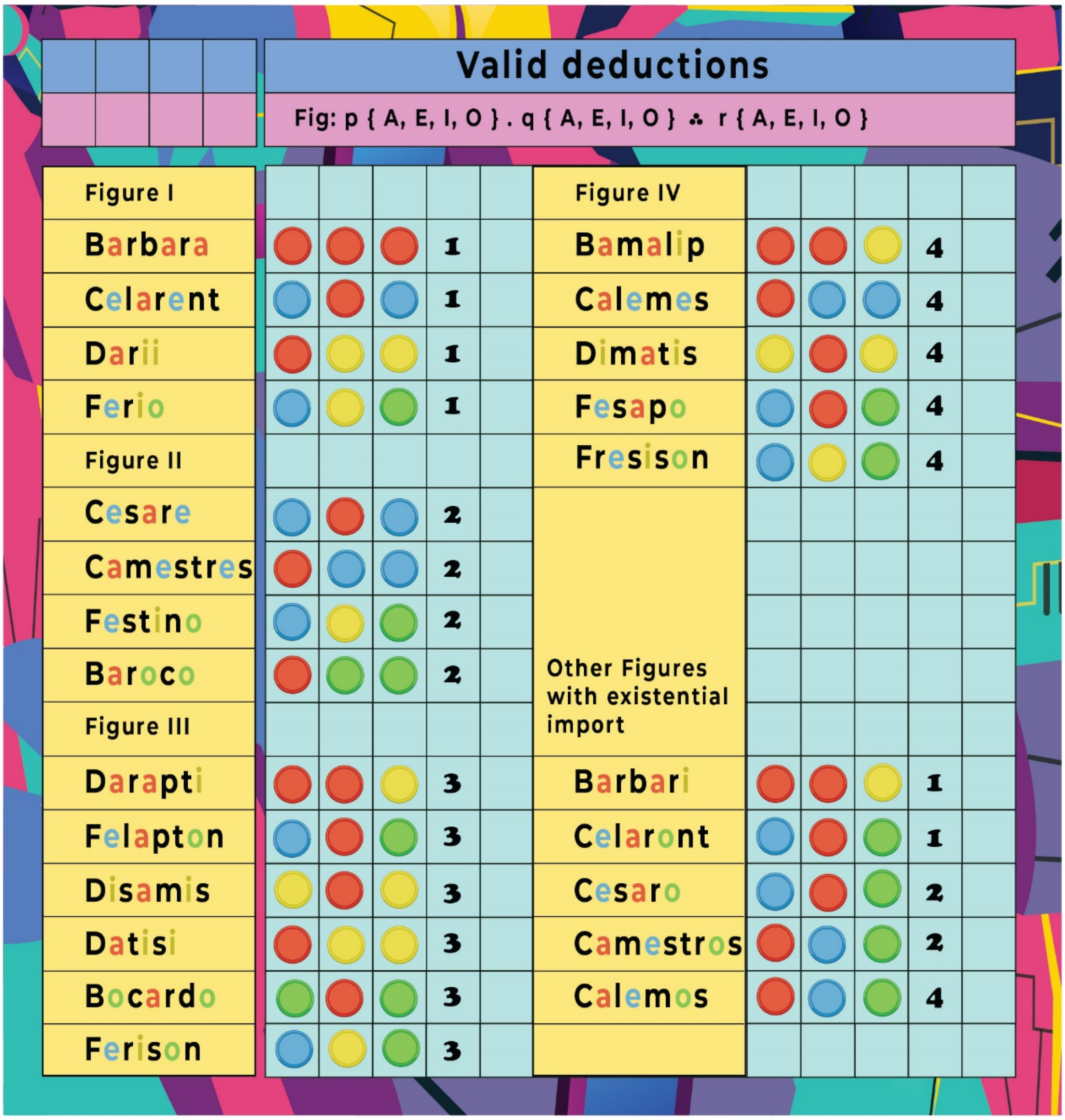
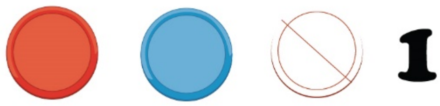


TABLE 93

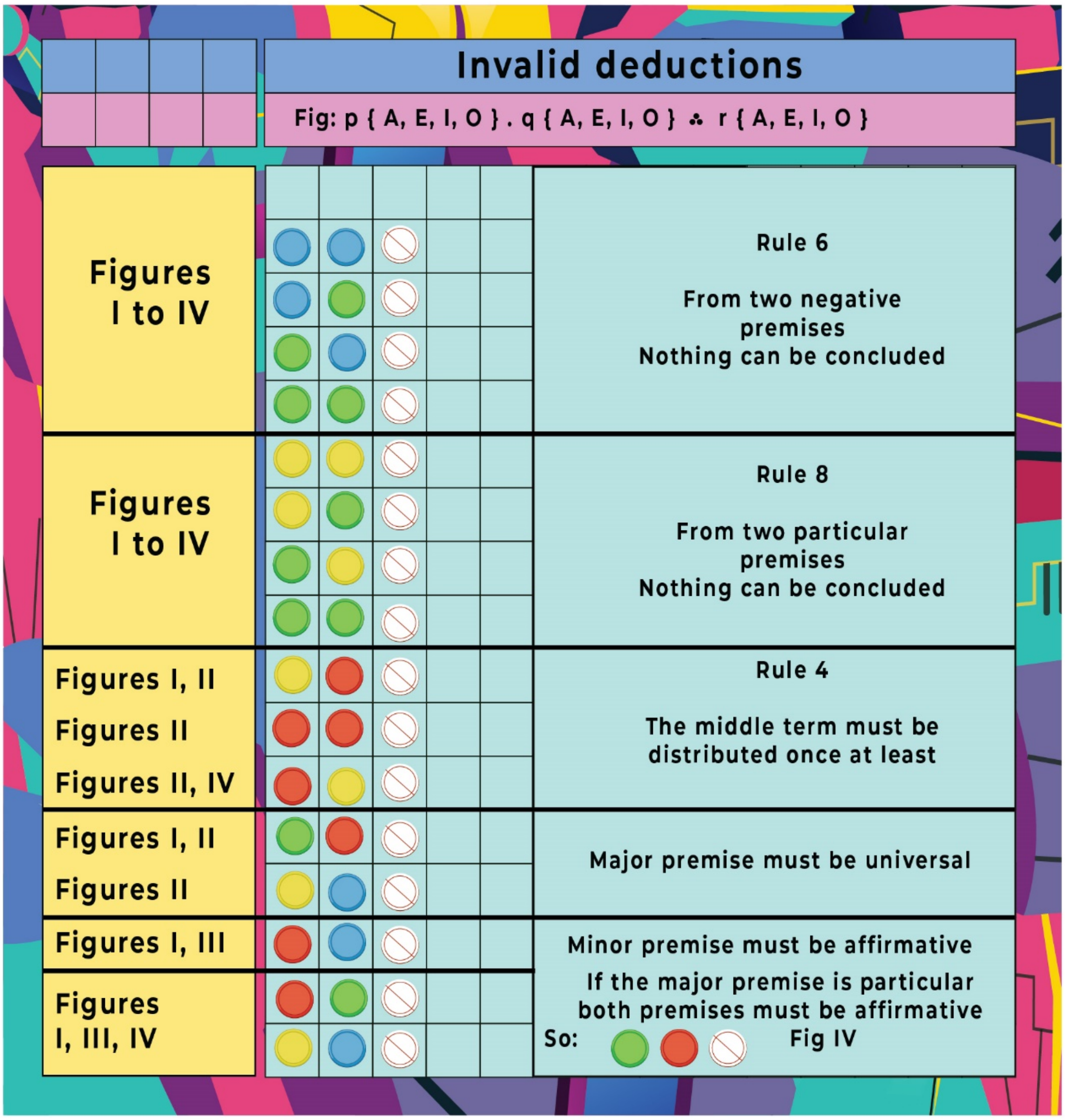
INVALID REASONING MODELS NOT IN CONFORMITY WITH THE RULES OF LOGIC

The following table illustrates, among the 256 possible categorical reasoning models, the forms of invalid reasoning. For example, the equation whose premises are of the form **EE** will be transcribed into the game as follows:



The sign  indicates that the conclusion is invalid. In this example, according to Rule 6, the reasoning is not correct because 'from two negative premises, nothing can be concluded'.

TABLE OF POSSIBLE PAIRS OF PREMISES SHOWING WHAT RULE FORBIDS THE DRAWING OF ANY CONCLUSION



On the pedagogical level of learning logic, children will be able to deduce the conclusion of categorical syllogisms from the eight rules outlined above, some of which have already been mentioned in the 'Battle game' of the Square of Opposition (Game 4). They will note, for example, the following rules: nothing can be deduced from two contradictory premises, from two negative propositions nothing can be concluded, from two particular propositions, a universal proposition cannot be deduced, the terms should not have more extension in the conclusion than in the premises, the middle term must be at least once universal (distributed) in the premises in order to be able to link them together. For example, the last two rules (No. 2 and 4) lead to fallacious reasoning, regardless of which words or phrases are written in place of the letters S, M, P[150].

[150] Sidgwich, A., 1914. *Elementary logic*. Cambridge: University Press. Reprint 2015, p. 53. The terms S, M, P are used here instead of X, Y, Z.

TABLE 94

EXAMPLES OF FALLACIES INVOLVED IN THE FOLLOWING SYLLOGISMS

AEE in Figure I

All M is P

No S is M

No S is P

Illicit process of the major (Rule 2)

Because P is distributed (universal term) in the conclusion, but undistributed (particular term) in the major premise.

AAA in Figure II

All P is M

All S is M

All S is P

Undistributed middle (rule 4: M is undistributed, particular, in both premises)

OAE in Figure III

Some M is not P

All M is S

No S is P

Illicit process of the minor (Rule 2)

IEO in Figure IV

Some P is M

No M is S

Some S is not P

Illicit process of the major (Rule2)

These rules and principles can be linked to the chromatic circle inscribed in the Enneagon of the nine categories of criteria proposed. For example, the result of 24 valid syllogisms meets the three categories of criteria listed on the Enneagon: 1, 4, 8. It illustrates how to conceive valid reasoning (in particular, the understanding criterion 1). It leads to a valid logical conclusion (the logical criterion 4). In the same way that mathematicians judge that some demonstrations are more elegant or aesthetic than others (the aesthetic criterion 8), Aristotle considers the valid syllogisms and conclusions of Figure I to be perfect. The other Figures are for him imperfect and can be reduced to the first Figure.

7.2.4 Using visual arts to visualise compound syllogisms

The models of syllogisms proposed by the Stoics introduce complex reasoning that requires the use of different connectors between premises (and, or, implies, etc.). The idea of Boole's logic and Truth Tables is to make the resolution of syllogisms mechanical by a method of calculation, called Calculus of syllogisms.
From the logic of the Stoics, there are five forms of valid syllogisms called The Five Undemonstrable arguments in the sense that they are obvious enough not to have to be demonstrated. Their improper use makes it possible to detect illicit reasoning. In Game 7, the application of Boolean logic and Truth Tables is used to determine the validity or invalidity of these Stoic forms of reasoning. For example, the player gains 5 points for the correct answer to a *modus ponens*. He loses 5 points (2 variables, p and q, and three signs or operators: =>, •,) for an incorrect answer.

TABLE 95

THE FIVE UNDEMONSTRABLE VALID SYLLOGISMS OF CHRYSIPPUS

Name	Example	Symbolic form
Modus ponens	If it is day, it is light. It is day. Therefore, it is light.	((p => q) . p) => q
Modus tollens	If it is day, it is light. It is not light. Therefore, it is not day.	((p => q) . ~q) => ~p
Modus ponendo tollens (conjunctive)	It is not both day and night. It is day. Therefore, it is not night.	(~(p . q) . p) => ~q
Modus ponendo tollens (disjunctive) (conjunctive)	It is day or night. It is day. Therefore, it is not night. It is not both day and night. It is day. Therefore, it is not night.	((p ⊕ q) . p) => ~q (~ (p .q) . p) => ~q
Modus tollendo ponens (disjunctive)	It is either day or night. It is not day. Therefore, it is night.	((p ⊕ q) . ~p) => q

TABLE 96

LIST OF SYMBOLS USED IN THE GAME

Variables: p, q (propositions p, q)

AND •
(And, yet, but, however, moreover, nevertheless, still, also, as well as, whereas, while, although, both, additionally, furthermore, or, comma, semicolon).

NOT ~
(Not, no, non, it is not the case, it is false that)

Or inclusive +
(Or, Either ... or...)

Or exclusive ⊕
(One or the other but not both at the same time)

Material implication ⇒
(If ... then, implies, given that, in case, provided that, on condition that, sufficient condition for, necessary condition for)

Valid conclusion ∴
(Meaning 'this is the conclusion', and this conclusion is valid (or correct in common language).

Invalid conclusion
(Meaning the conclusion is invalid (or wrong in common language).

A fallacy is an error of reasoning. The purpose of the games is both to show how to establish correct reasoning but also to be able to denounce fallacious reasoning. The games illustrate through the visual arts several forms of logical error in Stoic compound syllogisms. Tradition has given names to the most common forms of invalid arguments: *Fallacy of affirming the consequent* in the *modus ponens*, *Fallacy of denying the antecedent* in the *modus tollens*, *Fallacy in the disjunctive syllogism*, etc. In the following example of the fallacious *modus ponens*, the conclusion does not necessarily follow the premises: it can be night and light with electric light. One source of error in the disjunctive syllogisms is the confusion between inclusive and exclusive 'or'. The 'inclusive or' (noted +) has three possibilities, while the 'exclusive or' (noted ⊕) has only two possibilities: it is either one thing or the other, but not both at the same time.

TABLE 97

SOME COMMON INVALID ARGUMENT FORMS IN STOIC COMPOUND SYLLOGISMS

Name of fallacies	Example	Invalid symbolic form
Fallacy of affirming the consequent	If it is day, it is light. It is light. Therefore, it is day.	((p => q) . q) => p
Fallacy of denying the antecedent	If it is day, it is light. It is not day. Therefore, it is not light.	((p => q) . ~p) => ~q
Modus ponendo tollens (disjunctive) Fallacy in disjunctive syllogism	It is day or night. It is day. Therefore, it is not night.	((p + q) . p) => ~q
Fallacy in disjunctive syllogism *Modus tollendo ponens* (disjunctive)	It is day or night. It is not day. Therefore, it is night.	((p + q) . ~p) => q

In Game 7, the application of Boolean logic and Truth Tables is used to determine the validity or invalidity of these Stoic forms of reasoning. For an incorrect answer, the player loses 5 points (2 variables, p and q, and three signs or operators: =>, •,). He gains 5 points for the correct answer to a modus ponens.

The game 7.1 allows to establish a bridge between Aristotle's logic and the logic of the Stoics, before using Boole's logic and the Truth Tables.

7.2.5 The visual conversion of compound syllogisms into categorical syllogisms

It is possible here to switch from one mode of reasoning to another. By using the visual arts, an interesting result is achieved. On the one hand, one can better perceive Aristotle's point of view on compound syllogisms and, on the other hand, one can nuance Peter Kreeft's point when he writes about the logic of the Stoics (2004, ed. 2014, p. 289): 'We cannot use Euler's circles, Aristotle's six rules, Venn diagrams, or "Barbara Celarent" in checking them. They do not have mood or figure. They do not have major, minor, and middle terms, or major and minor premises.' However, if one uses counters of various shapes and colours, one can perceive why Aristotle proposed to convert compound syllogisms into categorical syllogisms. The middle term becomes a whole proposition, and this proposition allows to move from the premises to the conclusion. As with the middle term, this proposition disappears from the conclusion, as shown in the following figures.

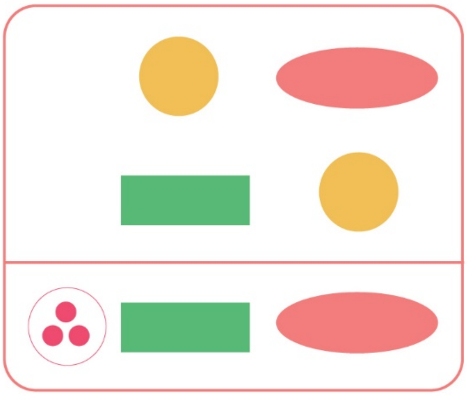
TABLE 98

ARISTOTELIAN SYLLOGISMS (FIGURE 1, BARBARA AND CELARENT MOODS)

Two illustrated examples using circles, ellipses and rectangles and the negation sign and the conclusion

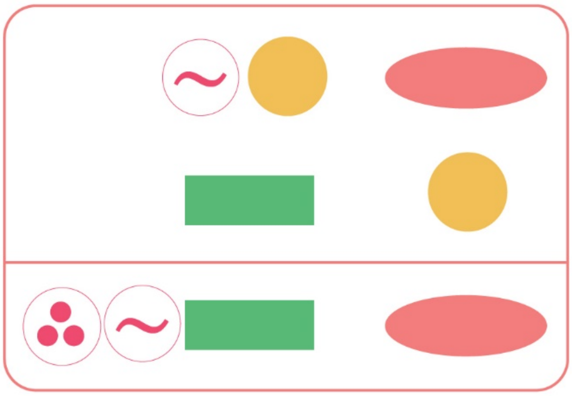
Barbara

All M is P
All S is M
Therefore, all S is P



Celarent

No M is P
All S is M
Therefore, no S is P



The visual approach shows the disappearance of the middle term in the conclusion.

TABLE 99

STOIC SYLLOGISMS (MODUS PONENS ET MODUS TOLLENS)

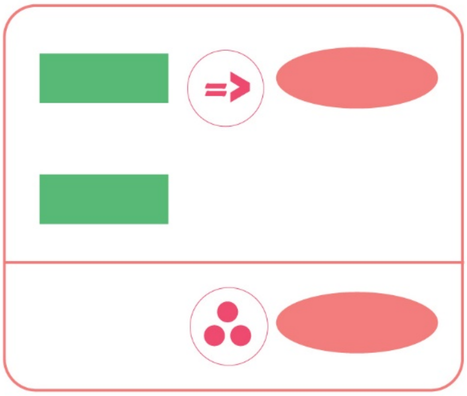
Two illustrated examples using circles, ellipses and rectangles and the negation sign and the conclusion

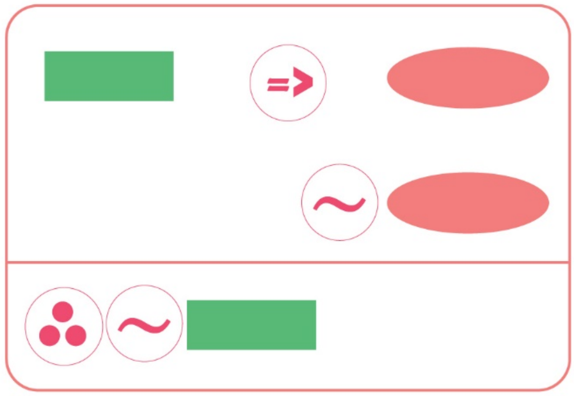
Modus ponens

If p then q
p
Therefore, q

Modus tollens

If p then q
not q
Therefore, not p





In summary, these diagrams which I have shaped into circles, ellipses and rectangles in the way Byrne (1847) visually solves equations show two things:

In Aristotle's logic, the two premises of the syllogism play the role of antecedent and are wholly contained in the conclusion (the consequent).

In Stoic logic, the conclusion of the syllogism is entirely contained in the conditional (or hypothetical) main premise. For example, in the *modus ponens*, the first part of the proposition (the red rectangle) disappears from the conclusion. In the *modus tollens*, this is the second part of the proposition (the green oval) that disappears from the conclusion.

Because of the hypothetical (If… Then) contained in the modus ponens and pollens, one could think that these syllogisms apply to experimental sciences, whereas Aristotle's categorical syllogisms would only apply to pure sciences without hypotheses in the premises.

However, the way of presenting the difference between the syllogisms of Aristotle and the Stoics using colours allowed me to solve a paradox I had been wondering about. If a distinction is made between pure deductive sciences and empirical inductive sciences, how did Aristotle, who is often regarded as an empiricist[151], manage to lay the foundations of pure logical deductive sciences for generations to come? By considering the difference between truth and validity of an argument, the paradox of Aristotle's empiricism disappears. My interpretation, which has been useful for my drawings and for building in Game 7 a bridge between storytelling and pure science is this. In a syllogism, truth concerns the premises and the conclusion, while validity – the object here of the illustrations – concerns reasoning. In the experimental sciences, in contrast to pure mathematics, the premises are often established empirically. However, this does not prevent correct reasoning and carrying out thought experiments that often have the characteristics of a fairy tale. Even among the Stoics, for the modus ponens (if p then q, and p) or modus tollens (if p then q, and not-q) to work, one must respectively affirm the antecedent (p) or deny the consequent (not-q). This kind of affirmation or denial could be fairy tales (true on *our planet* or on *another planet*, such as Lewis Carroll says).

Jules Tricot (1893–1963), known for his translations of ancient authors, whose Aristotle's *Organon,* shows how to pass from the syllogisms of the Stoics to the categorical syllogisms of Aristotle. In the following two examples, it is true that the middle term, replaced by a sentence, hasn't completely disappeared from the premises. Furthermore, the hypothesis barely announced in the first premise is immediately affirmed categorically in the second premise. As Tricot shows the *modus ponens* can be reduced to Barbara (AAI), a reduction that was made by Aristotle.

[151] Kaye writes (2008, reprint 2017, pp. 24–25): 'Aristotle is an empiricist, maintaining that knowledge comes from observation of the world' while 'Plato rejected empiricism because he regarded the senses as untrustworthy'. Hence the question I asked myself: is there so much difference here between Aristotle who is a student of Plato? There is probably little difference here.

TABLE 100

MEDIEVAL LOGIC: BARBARA, CELARENT, CAMESTRES...

The *modus tollens* can reduced to Barbara (AAA1)

Modus ponens	Barbara
If S is A, then S is B.	SA is SB.
S is A.	S is SA.
Therefore, S is B	Therefore, S is SB.

And the *modus tollens* can reduced to Camestres (AEE2)

Modus tollens	Camestres
If S is A, then S is B.	SA is SB.
S is not B.	S is not SB.
Therefore, S is not A.	Therefore, S is not SA.

Similarly, Aristotle's logic proposes to reduce disjunctive syllogisms (*modus ponendo-tollens* and *tollendo-ponens*) of the form: S is A or B into categorical syllogisms (Celarent and Barbara), and to reduce into Celarent (EAE1) the Chrysippus' *modus ponendo tollens*, i.e. the conjunctive syllogism based on an incompatibility (Tricot, 1973, p. 234):

TABLE 101

REDUCTION OF DISJUNCTIVE SYLLOGISMS INTO CATEGORICAL SYLLOGISMS

Chrysippus' conjunctive syllogism	Aristotle's Celarent syllogism
It is not both night and day (S is not A and B).	SA is not SB.
It is day (S is A).	S is SA.
Therefore, it is not night (S is not B) .	Therefore, S is not SB.

Note: the copulating syllogism is first reduced to a hypothetical syllogism: S is not A and B is reduced to an equivalent hypothetical syllogism where the major is negative meaning: If S is A, S is not B, then to its whole this syllogism is reduced to a categorical syllogism called Celarent. In other words, the disjunction 'or' is an exclusive disjunction: either one or the other, but not both at the same time.

However, given the difficulties of reducing compound syllogisms to categorical syllogisms, it is simpler to check their validity by using the Truth Tables mechanically as proposed in Game 7.2.

7.2.6 Using visual arts to solve compound syllogisms by means of Truth Tables

From the compound syllogism model, the Game 7.2 makes it possible to test the validity of different short stories: hypothetical syllogism, dilemmas, contradictory arguments in a police investigation, *reductio ad absurdum*, etc., using Truth Tables. It consists of a game board and counters. The truth-table method is based on the concept of enumerating all possible cases. The construction of a Truth Table is done in a three-step visual game: firstly, combining the truth values of the premises (noted true: T, false: F), secondly, defining a logical operator to characterise the conclusions obtained (and, or, implies, etc.), thirdly, determining the Truth Table of each logical operator (and, or, implies, etc.) obtained from two premises (called the truth table of logical functions or operators F1 to F16).

TABLE 102

FIRST STEP: COMBINING THE TRUTH VALUES OF THE PREMISES

p / q	True	False
True	T, T	T, F
False	F, T	F, F

A syllogism is composed of 2 premises, p and q, and a conclusion noted C. The 2 premises, p and q can be true (T) together, or they can be false (F) together, or p can be true and q false, or the reverse p can be false and q true. Hence, there are $2^2 = (2 \times 2) = 4$ possibilities as indicated in the following cross-table.

p	q
T	T
T	F
F	T
F	F

First steps. This cross table can be written in columns where the number of rows is equal to $2^2 = 2 \times 2 = 4$ rows.

With three propositions (p, q, r), there will be $2^3 = 2 \times 2 \times 2 = 8$ possible enumerations, i.e. 8 rows, with 4 propositions: $2^4 = 2 \times 2 \times 2 \times 2 = 16$ rows, and for n propositions (p, q, r, s, t, etc.), there will be 2^n possible enumerations, i.e. 2^n rows.

TABLE 103

SECOND STEP: DEFINING A LOGICAL OPERATOR TO CHARACTERISE THE CONCLUSIONS OBTAINED FROM TWO PREMISES.

The conclusion (noted C) of a syllogism can be true (T) or false (F). For example, let us consider the possibility of having the following conclusion C1:

p	q	C1
T	T	T
T	F	F
F	T	F
F	F	F

This table indicates that there is only one case where the conclusion is true (T). This case is realised when both premises p and q are true. This table corresponds to the idea that logicians have of the AND connector that links two propositions p and q.

Let us consider another possible conclusion C2.

p	q	C2
T	T	T
T	F	T
F	T	T
F	F	F

This table indicates that there are three cases where the conclusion is true (T). These cases are realised when both premises p and q are true or when one of the two premises is true. This table corresponds to the idea of the non-exclusive OR connector (also called inclusive OR) that links two propositions p and q.

The third step is to list all possible conclusions from two premises p and q that may be true or false, one true, the other false and vice versa. It was this set of combinations that Wittgenstein established in the *Tractatus Logico-Philosophicus* (1922, point 5.101), which he presented at Russell's request as a doctoral thesis at Cambridge. By combining with order and repetition the two letters T and F of the 2^2 input combinations of the two propositions p and q, one can exhaustively count 16 types of operators or conclusions C1, C2, C3… C16 ($4^2 = 4 \times 4 = 16$). This table makes it possible to rediscover the meanings of the principal connectors (and, or inclusive, or exclusive, implies, etc.) defined in the traditional logic of Aristotle and the Stoic logic in particular, essentially: the negation 'Not', the conjunctive proposition 'And', the disjunctive proposition 'Or', exclusive or inclusive, the hypothetical proposition: If p then q (noted $p \Rightarrow q$). Boolean logic will transform the Truth Tables into a calculator table. To do this, a true proposition, denoted p = T, is symbolised by 1 in Boolean logic: p = 1, and a false proposition, p = F, by p = 0, with convention: 1 and 1 = 1. These operators (or connectors) are frequently referred to as the function between the two variables p and q.

TABLE 104

TRUTH TABLE OF LOGICAL FUNCTIONS (F1 TO F16)

p	q	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13	F14	F15	F16
p	q	T	OR	$q \Rightarrow p$	p	$p \Rightarrow q$	q	$\leftrightarrow$	AND	NAND	XOR	Not-q	Not $(p \Rightarrow q)$	Not-p	Not $(q \Rightarrow p)$	NOR	$\perp$
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

In modern logic, each truth function (F1 to F16) now has a name, a symbol and a precise meaning. The AND conjunction in everyday language can no longer be confused with the logical AND connector or conjunction, symbolised according to the authors by different signs: &, ∧, '.'

Meaning of connectors or logical operators:

The main operators are:
AND (Col. 8). The logical conjunction p AND q is also written as $p \wedge q$, or p.q. The two propositions are true together: p AND q if and only if p is true at the same time as q.

OR (Col. 2). The inclusive disjunction p OR q is also written as $p \vee q$, or $p + q$. The disjunction is true three times: if p is true, if q is true or if p and q are true.

XOR (Col. 10). The exclusive disjunction p XOR q is also written as $p \oplus q$. It means: either p or q, but not both.
p => q (Col. 5). This is the traditional logical implication, called 'material implication': $p \Rightarrow q$. If p implies q (noted if p, then q). When p is true and q is true, the conditional is true. When p is true and q is false, the conditional is false. The problem arises when the antecedent (p) is wrong. The traditional logic considers that from the false it is possible to deduce anything (ex *falso sequitur quodlibet*) and therefore that the conclusion (p implies q) is always true. This condition can be overcome in non-conventional logic by rejecting as true conclusion q if the antecedent p is false.

p < – > q, or XNOR (Col. 7). The logical equality or the bi-implication is noted $p \leftrightarrow q$. It is true if and only if p and q are true, or if p and q are false at the same time. It reflects the idea of a 'necessary and sufficient condition' where p is 'equivalent' to q.

NEGATION (Col. p and 13, q and 11). It is the most common operator which is not a connector between two propositions. If p is true, the negation (non-p) is false and vice versa, if p is false, the negation (non-p) is true. It is the same for q.

The least used connectors are:
Col. 9. The logical NAND or the incompatibility is noted with Sheffer's bar p\q, or written as $p \uparrow q$. It produces a value of false if both of its operands are true.
Col. 3. The implication goes from q to p (and not from p to q, column 5). It translates expressions such as: if q then p, q only if p, p provided that q.
Col. 4 and 6 respectively mean: p independently of q, p regardless of q; or q independently of p, q regardless of p.
Col. 12 and 14 respectively mean the negation of column 5, i. e. [not (p – > q)] formula equivalent to (p AND not-q) and the negation of column 3; i. e. [not (q – > p)] equivalent to (q AND not-p).
Col. 15. The logical NOR means 'neither p nor q'. This is the negation of column 2, which is noted with the arrow down, called the Pierce arrow '$p\downarrow q$' of the American logician Pierce (1839–1914), meaning that it produces a value of true if both of its operands are false.
Col. 16. The contradiction is noted $\perp$.

These operators are represented in the game by cards.

Several functions of the truth table are simple. The function F1, tautological is equal to 1. The function F16 which expresses the contradiction is equal to 0. Functions F2, F8, F11 and F13 correspond to the elementary operators inclusive OR, AND, Negation. All the other functions, which are more complex, can be expressed by means of the elementary operators (table 10). This makes it possible in Games 7.2 and 7.3 to transform short stories into logical equations.

TABLE 105

EQUIVALENCE OF COMPLEX FUNCTIONS EXPRESSED USING ELEMENTARY OPERATORS

Negation is indicated here by: p' = non-p ; q' = non-q.

F1 = 1	F5 = p' + q	F9 = p' + q'	F13 = p'
F2 = p + q	F6 = q	F10 = pq' + p'q	F14 = p'q
F3 = p + q'	F7 = pq +p'q'	F11 = q'	F15 = p'q'
F4 = p	F8 = pq	F12 = pq'	F16 = 0

Examples: F10 = exclusif OR = pq' + p'q ; read: [(p and not-q) or (not-p and q)].
F5 = material implication p ⇒ q is equivalent to p' + q (read: not-p or q).
F15 = neither p nor q = p'q' (read: not-p and not-q)
Comments. Here, the logical symbols + and the dot '.' do not correspond exactly to the same symbols used in arithmetic. However, the concept which animates Boole's logic is to get as close as possible to an algebraic method which deals not with numbers but with propositions p, q, not-p (p'), not-q (q'), etc.

EXAMPLE

OF THE APPLICATION OF THE TRUTH TABLE

Put in symbolic form, players can verify this dilemma:

If p, then q; and if r then s	((p => q) . (r => s))
Either p or r	p + r
Therefore, either q or s	q + s

What is written with the cards of the game:
[((if p => q). (if r => s)) . (p + r)] ∴ q + s

or, the conjunction 'And' (noted, '.') being commutative:
[(p + r) . ((if p => q) . (if r=> s))] ∴ q + s

Truth Table Proof of Complex Constructive Dilemmas
[((if p => q) . (if r => s)) . (p + r)] ∴ q + s

p	r	q	s	[[(p	=>	q)	AND	(r	=>	s)]	AND (1)	(p	OR	r)]	(1) => (2)		q Or s (2)
Col. 1	Col. 2	Col. 3	Col. 4		Col. 5		Col. 6		Col. 7		Col. 8		Col. 9		Col. 10		Col. 11
1	1	1	1		1		1		1		1		1		1		1
1	1	1	0		1		0		0		0		1		1		1
1	1	0	1		0		0		1		0		1		1		1
1	1	0	0		0		0		0		0		1		1		0
1	0	1	1		1		1		1		1		1		1		1
1	0	1	0		1		1		1		1		1		1		1
1	0	0	1		0		0		1		0		1		1		1
1	0	0	0		0		0		1		0		1		1		0
0	1	1	1		1		1		1		1		1		1		1
0	1	1	0		1		0		0		0		1		1		1
0	1	0	1		1		1		1		1		1		1		1
0	1	0	0		1		0		0		0		0		1		0
0	0	1	1		1		1		1		0		0		1		1
0	0	1	0		1		1		1		0		0		1		1
0	0	0	1		1		1		1		0		0		1		1
0	0	0	0		1		1		1		0		0		1		0

Number of variables: 4
Number of lines: $2^n = 2^4 = 2 \times 2 \times 2 \times 2 = 16$
Col.6, AND: ((if p => q) . (if r => q))
Col. 8, AND: col.6 and col.9, i.e. [((if p => q) . (if r => q)) . (p + r)]
Col.10, implies: col. 8 => col. 11: tautology
It can also be noted that the problem of confusion between the implication (if … then: col. 8 => col. 11) and the term that announces the conclusion (therefore, thus, col.10) – and which is the subject of Lewis Carroll's Achilles' paradox and the Tortoise – is solved here by a system of equivalence between two parts of a logical equation (col. 8 and col. 11) noted in col. 10 by the number 1.

The formula leads to a tautology (all truth values are equal to 1); therefore, the formula is valid.
[((if p => q) . (if r => s)) . (p + r)] ∴ q + s

This formula is reduced to a simple dilemma if s = q, because q + q = q (tautology).

7.2.7 Using visual arts to bridge the gap between logic and 'The Electricity Fairy'

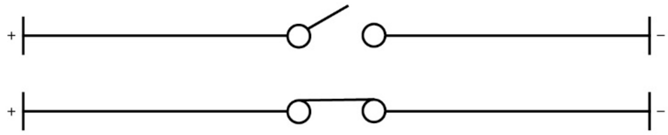
Parallel to the symbolic development of logic, and without any direct link, the construction will give the main connectors (and, or) a physical and concrete image. There are educational games for children, from the age of 8, which allow them to carry out simple experiments to discover the world of electricity. To bridge the gap between logic and the physical sciences, I use the elements supplied in two game boxes (Clementoni, 2008) to make a few electrical circuits from a 3-volt battery (2 x 1.5 volts)[152] to visualise the two logical operators: And, OR.

[152] Clementoni, 2008. *L'Électricité* (languages: French and Italian, 8 + years, with a 32-page booklet), La Chapelle-sur-Erdre, France : Le Labo des curieux, Clementoni.

TABLE 106

VISUAL APPLICATION OF LOGIC CONNECTORS AND, OR

Electrical diagram

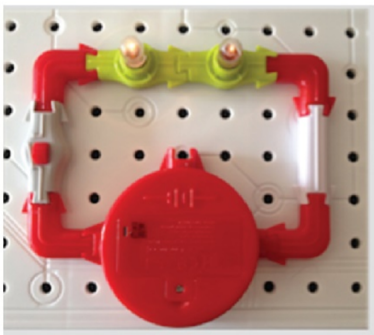


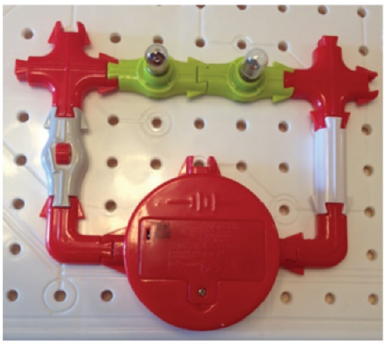
P open = 0

P closed = 1

By convention, if p is the switch:
p = 1 (means p is closed = ON): current is flowing.
p = 0 (means p is open = OFF): current does not flow.

First: Connector And
Electrical circuit in series: the switch is ON



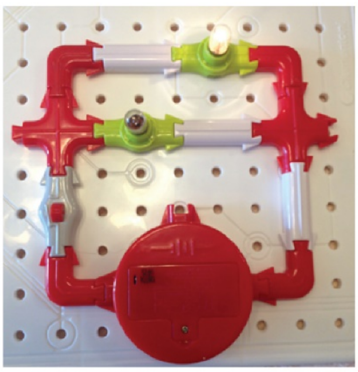


p	q	C1
1	1	1
1	0	0
0	1	0
0	0	0

p = 1, q = 1 p = 0 or q = 0

The switch is ON: the current is flowing, but in a 'series' electrical circuit, if one bulb (p or q) is damaged or removed, the other bulb also switches off. This corresponds to the truth table of the AND-operator of the two lamps p and q (1 + 1 = 1, 1 + 0 = 0, 0 + 1 = 0, 0 + 0 = 0).

Second: Connector Or (non-exclusive)
Lamps connected in 'parallel'.
The switch is ON, but a bulb is damaged or removed.



p	q	C2
1	1	1
1	0	1
0	1	1
0	0	0

In a parallel electrical circuit, if one bulb is damaged or removed, the other bulb stays lighted. This corresponds to the truth table for the Or non-exclusive operator of the two lamps p and q (1 + 1 = 1; 1 + 0 = 1; 0 + 1 = 1; 0 + 0 = 0).

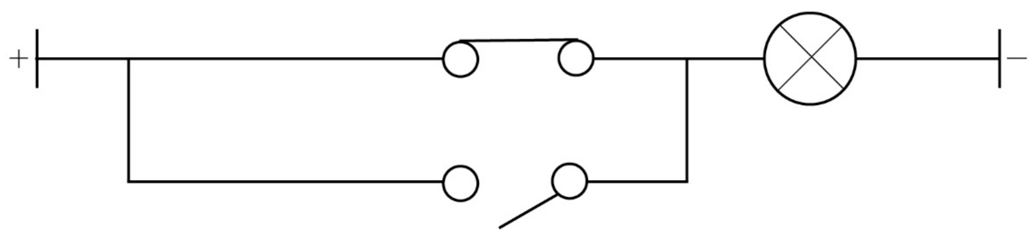
7.2.8 Using visual arts to bridge the gap between logic and Electronics

The introduction of electronic components (transistors, diodes, semiconductor diodes, etc.) allowing the passage of the electric current in one direction, while blocking it in another direction, will promote the use of Truth Tables in computing. The following example allows representation of the material implication (Function F5 of the Truth Table).

TABLE 107

EXAMPLE OF A DIAGRAM OF AN ELECTRICAL CONTACT SCHEME: THE MATERIAL IMPLICATION (=>)

Contact scheme: p implies q is equivalent to non-p or q.

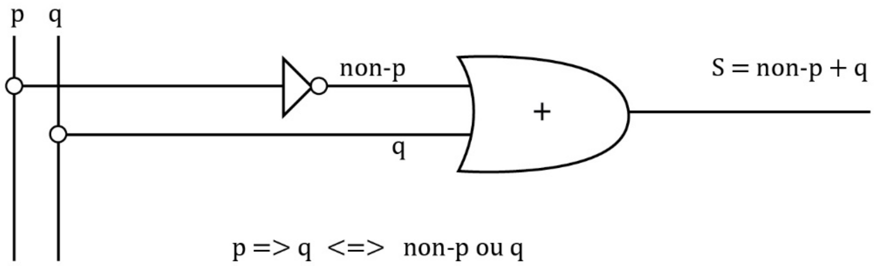


New symbols will appear to build logigrams, as in the following diagrams of the equivalence of implication and the exclusive Or, noted XOR.

TABLE 108

LOGIGRAM: P IMPLIES Q IS EQUIVALENT TO NON-P OR Q

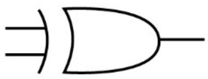
The first symbol reverses the statement p to not-p. The second symbol logically adds the proposition q to the previous result.

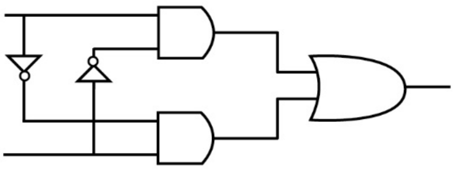


The exclusive disjunction p XOR q, also written in Game 7 as p ⊕ q means 'either p or q, but not both' is often represented as follows. It is this operator that is used in Stoic dilemmas.

TABLE 109

XOR DISJUNCTION (EXCLUSIVE)

<table>
<tr><th>FUNCTION</th><th>SYMBOL
XOR = exclusive Or = $\oplus$</th><th>EQUATION</th><th>TRUTH TABLE</th></tr>
<tr><td>XOR</td><td>


</td><td>S = p $\oplus$ q

Equivalent formula (F10)

S= pq’ +p’q</td><td><table><tr><th>p</th><th>q</th><th>S</th></tr><tr><td>1</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>0</td><td>0</td><td>0</td></tr></table></td></tr>
</table>

7.2.9 Using visual arts to bridge the gap between logic, coding and computer science

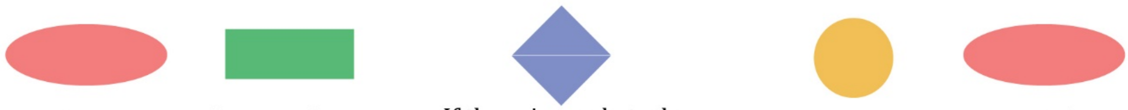
The combination of electrical and electronic systems with Truth Tables and Boole’s binary algebra has enabled the development of computer science. In recent years, several game publishers have been offering young children an introduction to computer coding and programming[153]. For example, children are able to programme a robot that moves according to the instructions given on the squares of a draughtboard: ‘Down, Right, Right, Up, Left, End.’ The Game 7, called the Robot, highlights two important instructions used in programming based on formal logic: the modus ponens (If … then) and the concept of repetition (loop). This can be symbolised by the following diagram. Here, the Robot must go from one point to another, and go around an obstacle in its path. The points are generally identified by the Cartesian coordinate system, x, y, which children must code.

[153] Examples. Prottsman, K., 2017. *My First Coding Book. Packed with flaps and lots more to help you code without a computer! Down, Right, Right, Up, Left, End.* London: DK, Penguin Random House, and Jeu éducatif, 2019. *J’apprends à coder*. Paris: Nathan.

TABLE 110

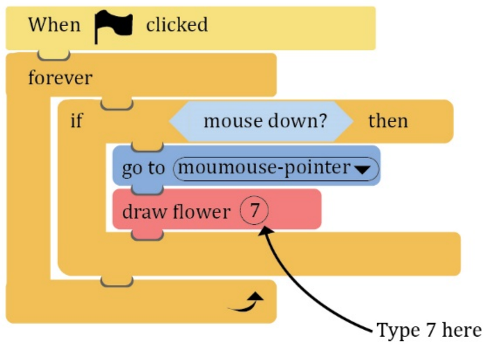
THE MODERN AGE OF PROGRAMMING AND CODING

For coding, children will use, for example, the 'Scratch language', whose *modus ponens* is represented as follows[154]:



Sart	Advance of a case	If there is an obstacle, then jump over it If thre is no obstacle then advance one square ahead	Loop 3: advance threes quares straight	End: arrival

The scracth language: an example



The computer will draw flowers with seven petals each.

[154] Vorderman, C. 2019. *Computer coding. Projects for Kids*. Ages 8–16. London: DK, Penguin Random House, p. 113, Art, Fantastic Flowers.

Bibliography

Preliminary notes

This biography consists of two parts. The first one indicates 215 references for books, articles and websites cited in the thesis. The fundamental and essential texts referred to, for example, Aristotle's *Organon*, La logique de Port Royal ou l'Art de penser ('The Logic of Port Royal or The Art of Thinking'), are referenced in English and French. It allows for a comparison of the translator's comments and interpretations, especially when texts are initially written in Ancient Greek or Latin.

There are several ways of referencing authors and their works. Some bibliographies indicate, for example, the surname and first name of the author as well as the number of pages of the works. This makes it possible to distinguish between a popular work of a few hundred pages and major work such as the *Organon* in VI volumes or the *Elements of Euclid* in IX books and 2 volumes. Here, the presentation adopted in the thesis is that recommended by Anglia Ruskin University (A.R.U. University Library website, April 2019): Harvard style of Referencing. Only the initials of the first names are indicated. The number of pages of the books is unspecified. In addition, to the fact that some renowned authors are generally cited either with or without their first name (e.g. Lewis Carroll, Bertrand Russell, Euclid, Descartes, Pascal), first names may be given in the in-text so as not to confuse, for example, John Maynard Keynes, a renowned economist, and John Neville Keynes, a little-known author nowadays referred to by Lewis Carroll. As recommended in the Harvard style of Referencing, the first year of publication is shown in the bibliography with the author's name, whenever this information is available[155]. This avoids writing in the in-text: Aristotle, 2012. On the other hand, to indicate the reference pages, the edition consulted is indicated in-text (Aristotle, ed. 2012, pp. x) or the full references can then be given in the footnote.

[155] The referencing model adopted in the bibliography is as follows: "For works which are reprints of classical original works, the reference should include details of the original date of the work and reprinting details, the suggested elements for such references being: Keynes, J. M., 1936. *The general theory of employment, interest, and money*. Reprint 1988. London: Palgrave Macmillan. An in-text reference for the previous example would read: (Keynes, 1936)". Extract from: A.R.U, University Library website. April 2019. *Guide to Harvard style of Referencing*. 6.1.2 Version, p. 26. Available through: Anglia Ruskin University Library website <library.aru.ac.uk> and available at: <https://library.aru.ac.uk/referencing/files/Harvard_referencing_201718.pdf> [Accessed 28 August 2020].

The second part of the bibliography concerns more specifically the cited illustrators and their works. It is employed for a case study (chapter 2.6) with statistical elements (the study of the image/text ratio, format of pop-ups and games, length x width x height, etc.). There may be minor differences here with the formats, the number of pages, number of illustrations, etc. indicated by the publishers. For statistical consistency and homogeneity, the figures given are those actually measured by a ruler and manually counted, from the books, pop-ups and games consulted.

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II- Bibliographical references relating to illustrations and illustrators, pop-ups, games and statistical ratios.

Table of contents

A) Bibliography of illustrations

1. Carroll, L., 1864. *Alice's Adventures Under Ground*. Reprint 2019 with a new jacket. London: The British Library. Noted version 1 in this thesis.

The tale is composed of 4 chapters and 37 drawings by the author. It includes 92 pages for the manuscript, 30 pages of comments, 1 page of legal notices and 5 blank pages, i.e. 128 pages which is a multiple of 8 (16 x 8 = 128). The format given by the distributor is 12.1 cm x 17.8 cm and the measured external format is approximately 12.8 cm x 19.2 cm, i.e. a ratio of 17.8/12.1 = 3/2 = 1.5.

2. Carroll, L., 1865. *Alice's Adventures in Wonderland.* Reprint 2006. London: Wordsworth Editions. Noted version 2.

The tale includes 42 illustrations by John Tenniel and 12 chapters, including 4 new ones compared to the original version. The 99-page text is put in a 15 cm x 22.8 cm format, a ratio of about 3/2. '*Alice's Adventures Under Ground*' contains 12,715 words compared to *Alice's Adventures in Wonderland* which was expanded by Carroll to 26, 211.

Sir John Tenniel (Bayswater, London, 28 February 1820 - 25 February 1914) was a British illustrator, best known for his illustrations of Lewis Carroll's *Alice's Adventures in Wonderland*. He was one of the first to represent Santa Claus in 1850.

He also drew numerous cartoons for *Punch magazine* in the late 19th century. He first trained on his own, before entering the Royal Academy.

3. Carroll, L. 1890. *The Nursery 'Alice'*, Illustrated by Sir John Tenniel. The book was first published in 1890 by Macmillan, 25 years after the original *Alice's Adventures in Wonderland* and featured a new illustrated cover by E. Gertrude Thomson. French translation, 2016. *Alice racontée aux petits enfants*. Illustrated by John Tenniel. Paris: Les Lutins de l'école des loisirs. Noted version 3.

The tale includes 20 of John Tenniel's illustrations from the original book coloured, enlarged and, in some cases, revised. It consists of 14 chapters.

In the French translation of the full text published by Les lutins de l'École des loisirs (2016), the illustrated text consists of 61 pages in a 15 cm x 19 cm format, i.e. a ratio of approximately 5/4 = 1.25.

The Nursery 'Alice' (1890) is a shortened version of Alice's Adventures in Wonderland (1865) by Lewis Carroll adapted by the author himself for 'children aged from Nought to Five'.

It is written as though the story is being read aloud by someone who is also talking to the child listener, with many interpolations by the author, pointing out details in the pictures and asking questions: 'look at the picture and you'll see how much Alice is growing up'.

4. Carroll, L., 1871. *Through the Looking-Glass and What Alice Found There*. Reprint 2006. London: Wordsworth Editions. Noted version 4.

The tale includes 51 illustrations by John Tenniel (including the chess game) and 12 chapters. In Wordsworth 2006, the story is 119 pages long and the format of the book is 15 cm x 22.8 cm, i.e. a ratio of about 3/2.

All four versions are by Lewis Carroll. Lewis Carroll was the pen name of Charles Lutwidge Dodgson, born 27 January 1832, died on 14 January 1898.

Other tales by Lewis Carroll have been studied more specifically in the context of the thesis: *The Game of Logic* (1886) and *Symbolic Logic, Part I. Elementary* (1896). These two books are not illustrated but contain diagrams similar to the Venn Diagrams.

5. Browne, A., 2015. Illustration of *Alice's Adventures in Wonderland.* London: Walker Books Ltd, first published by Julia MacRae Books, 1988, and published in 2015 by Walker Books.

The book includes 49 illustrations (excluding cover), 12 chapters and a 128-page editor, with portraits of Lewis Carroll, 123 pages of illustrated text, measured 19 cm x 28 cm. The external format is 19.5 cm x 28.5 cm, i.e. a ratio of about 3/2.

Anthony Browne, born in Sheffield, England in 1946, is a British author, draughtsman and illustrator. He is the author of children's albums. He studied graphic arts at Leeds College of Art, from which he graduated in 1967. He received the Hans Christian Andersen Prize for illustration (2000).

6. Jansson, T. M., ed. 2018. Illustration of *Alice's Adventures in Wonderland*. First publication of the illustrations in 1966. Stockholm, Sweden: Bonniers. Reprint 2018. London: Tate Publishing.

The book includes 64 illustrations, 12 chapters, 107 pages of illustrated text in 15.4 cm x 23.8 cm format, a ratio of about 3/2.

Tove Marika Jansson, born in 1914 in Helsinki and died in June 2001 in the same city, was a writer, painter, illustrator and comic-strip artist. In 1966, for her contribution as an author of children's books, she received the Hans-Christian-Andersen Prize. She studied at the Helsinki Academy of Fine Arts (1933–1937).

7. Ross, T., 2015. Illustration of *Alice's Adventures in Wonderland,* retold by Tony Ross. First published in 1993, London: Andersen Press Ltd. Reprint 2015.

The book includes 111 illustrations, 12 chapters, 94 pages in 19.6 cm x 26.4 cm format, a ratio of about 7/5.

Anthony Lee Ross, known as Tony Ross, born in 1938 in London, is a British illustrator and author of children's literature. He studied at the Liverpool School of Art, before embarking on a prolific career as an illustrator.

8. Saint Exupéry, A. de, 1943, French ed. 1999. *Le Petit Prince,* Paris : Gallimard, Folio pocket collection. The original edition, in French and English, was first published in the United States in 1943 by Reynal and Hitchcock, USA, Saint Exupéry being exiled to the United States from 1941 to 1943. The posthumous French edition was published by Gallimard in 1945. It is a poetic and philosophical works in the form of a children's tale. So begins the story: following an engine failure, the airman had to land in the Sahara Desert and try to repair his plane alone.

In the Folio Pocket edition, the book contains 47 illustrations by the author's hand, 27 chapters, 104 editor pages, 75 pages with a minimum of text, in a format of 10.8 cm x 17.8 cm, i.e. a ratio of about 8/5.

Antoine Marie Jean-Baptiste Roger, Comte de Saint-Exupéry, known as Saint-Exupéry (29 June 1900 – 31 July 1944), was a French writer, poet, journalist and pioneering aviator. He became a laureate of several of France's highest literary awards and also won the United States National Book Award. He is best remembered for his novella *The Little Prince* (Le Petit Prince) and for his lyrical aviation writing including Wind, Sand and Stars and Night Flight.

9. Sfar, J., 2019. Illustration and text based on the work of the *Little Prince* of Saint-Exupéry, Gallimard Jeunesse, Paris and CD recording 2018. Noted Sfar version 1.

The book includes 59 illustrations, 120 editor pages, 22.5 x 20 cm, 119 illustrated pages in Italian measured format: 22.9 cm x 20.5 cm, a ratio of about 1.11. And 59 full-page images are in the ratio 20.6 cm x 18 cm, about 1.14. The titles of the chapters are not indicated.

Joann Sfar, born in 1971 in Nice (France), is a French comic book author, illustrator, novelist and director. Author of numerous comic strips, he is a graduate of the École Nationale Supérieure des Beaux-Arts. Cartoonist, he is also a screenwriter and film producer.

10. Sfar, J., 2008. The comic strip by J. Sfar, based on the work of the *Little Prince* of Saint-Exupéry. Paris: Gallimard. Noted Sfar version 2.

The book includes 660 images (6 x 110 pages), 112 pages and 17 cm x 24 cm format publisher, 110 pages in 17.2 cm x 24 cm measured format, i.e. a ratio of about 7/5.

Statistical summary drawn up on the basis of the cited reference works (Table 1)

Table 1
Illustration ratios: summarised statistical information on the works cited

	Number images	Number chapter	Number pages	Ratio Image/chap.	Ratio Image/page	Format w	Format L	Ratio L/w
L. Carrol V1	37	4 (8)	92	9 (5)	40%	12.8	19.2	1.5
L. Carroll V2	42	12	99	4	42%	15.0	22.8	1.5
L. Carrol V3	20	14	61	1	33%	15.0	19.0	1.3
L. Carrol V4	51	12	119	4	43%	15.0	22.8	1.5
A. Browne	49	12	123	4	40%	19.5	28.5	1.5
T. Jansson	64	12	107	5	60%	15.4	23.8	1.5
T. Ross	111	12	94	9	118%	19.6	26.4	1.3
St-Exupéry	47	27	90	2	52%	10.8	17.8	1.6
J. Sfar V1	59	27	119	2	50%	20.5	22.9	1.1
Average (1–9)	53	15	100	5 (4)	53%	16.0	22.6	1.4
(V'1–9) N0 7	46	15	101	4 (3)	45%	15.5	22.1	1.4

Note. Col. 1 and 2: Illustrators, col. 3: number of images, col. 4. number of chapters, col. 5. number of pages, col. 6: ratio: images/chapters, col. 7: ratio images/pages, col. 8. ratio format (L/W), col. 9 and 10: external format, dimensions measure: length (L) and width (w).

B) Bibliography of pop-ups with a brief biography of the illustrators mentioned

Note. The format of the books and covers, the number of pages and images, the ratios have been calculated here from the books themselves in order to constitute homogeneous statistics.

1. Blackwell, Su and Fletcher, C., 2015. *The Sleeping Beauty Theatre*. London: Thames & Hudson Ltd.
The pop up includes 10 changeable scenes, 12 moveable characters, 5 free-standing props, 28 pages, format 22.7 cm x 28.0 cm, i.e. a ratio of 6/5. The pop-up is for children from 6 to 8 years old.

Su Blackwell (British, Born in Sheffield in 1975), received BA (Hons) Art & Design, Bradford College of Art and Design, UK, and received MA Textiles, Royal College of Art, London. She is an artist and art director. Her work includes advertising campaigns, music videos, and theatre set design.

Corina Fletcher is a designer who specialises in pop-up books and paper engineering. She trained in Graphic Design at Central St Martins School of Art and then in Visual Communication at the Royal College of Art, where she developed her passion for working in three dimensions and paper.

2. Bourgon, M., 2018. *Trois petits Indiens*. Vanves, France: Gautier-Langereau.

The pop up includes 10 pop-up images, unpaginated, 8 pages with cardboard cover in 18 cm x 25 cm format, i.e. a ratio of 7/5. Pop-up for children from 0 to 3 years old.

Mathilde Bourgon (born in 1985 in Besançon, France). She studied at the School of Decorative Arts in Paris. Has worked for 5 years as a designer motifs for African textiles, specialising in pop-ups.

3. Colombier, Chloé du, 2019. *Les contraires* (« The opposites »). Paris: Gallimard Jeunesse.

The pop-up book contents 21 pop-up images, 10 pages, format 17.5 cm x 17.5 cm, cover 18.1 cm x 18.1 cm, ratio 1/1. Pop-up for children from 1 to 3 years old.

Chloé du Colombier (born in Savoie, France), illustrator, from the Art Deco School in Strasbourg. She works in youth publishing (Gallimard Jeunesse, Casterman, Gulfstream, les éditions du Ricochet), the press (Bayard, Salamandre) and creative leisure (Poppik).

4. Duisit, B., 2018. *Hermès pop-up*, Paris: Actes-sud/Hermès, France.

The drawings are from the collections of the Hermès silk squares, and consist of 12 pop-up images, 28 pages, format 21 cm x 21 cm, i.e. a square ratio of 1/1.

Bernard Duisit, a paper engineer, is a great creator of French pop-ups whose books, paper theatres, are translated in many countries. He has a background in Art, Object and Graphic Design and is very interested in the world of paper, pop-ups and animated books.

5. Duprat, G., 2018. *Univers. Des mondes grecs aux multiunivers* (« Universe. From Greek worlds to multiuniverses »). Paris: Saltimbanque Éditions.

The book includes 21 pop-up images, 56 illustrated pages, in cardboard format 26.6 cmx 27.6 cm, i.e. a ratio approximately 1/1. 'A lively, scientific and poetic document' for children from 6 or 9 to 12 years old.

Guillaume Duprat (born in Paris in 1973) is a French author and illustrator. He is a graduate of École Estienne in Paris. As an 'independent researcher in cosmology', he invented the Cosmotron, an interactive game from the permanent exhibition at the Vaulx-en-Velin (Rhône) planetarium, which opened in January 2014.

6. Ehrhard, D., 2018. *9 jouets d'artistes* (« 9 artists' toys »). Paris: Albums, Les Grandes Personnes éditions. The book consists of 10 pop-up images, 24 illustrated pages, cardboard format 18.5 cm x 20.5 cm, i.e. a ratio of about 1/1. Pop-up for children from 0 to 3 years old.

Dominique Ehrhard, born in 1958 in Alsace, France, is a painter, writer and author of children's literature. He is the author of board games. After studying Fine Arts at the University of Strasbourg, he taught painting for several years in Morocco.

7. Fiorin, F., 2015. *Pop-up Haunted House*, written by Sam Taplin, designed by Matt Durber, paper engineering by David Hawcock and Keith Finch. London: Usborne Publishing Ltd.

The book includes 15 images pop-ups, 10 pages, format 22.5 cm x 28.4 cm, i.e. a ratio of about 7/5. Pop-up for children from 0 to 5 years old.

Fabiano Fiorin was born in Venice in 1964, where he still works. After completing his master's degree at the Venice State Art Institute, he began his career as a comic book designer and then progressed to illustrating children's books. He is a cartoon artist and illustrator. He works with leading international Publishing Houses as well as a painter, graphic artist and also works in visual communication.

8. Hawcock, D., 2019. *Leonardo da Vinci. Les incroyables machines. Paris:* Minui Jeunesse.
The book includes 6 pop-up images, 14 pages, 21 cm x 24 cm, external format 21.8 cm x 24.3 cm, i.e. a ratio of about 1/1. Pop-up for children from 6 to 8 years old.

David Hawcock is a British graphic designer. He is a paper engineer, author and illustrator of books for children and young adults.

9. Hess, P., 2009. *Peter Pan*. Retelling of J.M. Barrie's classic tale. London: Templar Publishing.
The book includes 9 images pop-up, 16 pages, format 26.3 cm x 31.0 cm, i.e. a ratio of about 6/5. Audiobooks. For children of all ages.

Paul Hess is an illustrator whose work has a surreal edge, his quirky style illustrates unusual children's books and adult fiction covers. He has illustrated two pop-out concertina books for Templar, The *Sleeping Beauty* (above) and *Little Red Riding Hood*.

10. Joffre, V., 2018. *Les saisons*. Paris: Gallimard Jeunesse.
The book includes 14 pop-up images, 10 pages, format 17.5 cm x 17.5 cm, i.e. a ratio of 1/1. Pop-up for children from 1 to 3 years old.

Véronique Joffre (born in 1982, is a French illustrator, graduated from the Estienne School and the École supérieure des arts décoratifs in Strasbourg (France). She is known for her illustrations in cut paper collage.

11. Lo Monaco, G., 2014. *Mrs Sonia Delaunay*. London: Tate Publishing.
The book includes 10 pop-up images, 17 numbered pages, format 15 cm x 20 cm, a ratio approximately 7/5.

Gérard Lo Monaco was born in 1948 in Buenos Aires. A poster designer and graphic designer, he was a decorator at Jérôme Savary's Grand Magic Circus. He ran his own company of string puppets and his wooden horse riding arena. He has created record covers, cover firsts and pop-ups. Model makers and artistic director of publishing, he founded his own graphic design studio in 1995, Les Associés réunis, located in Paris.

12. Newman, B., 2018. *Autour du monde*. (« Around the world »). Toulouse, France: Milan Eds.
The pop-up has 6 wheels to turn and 10 pages, format 24 cm x 28 cm, i.e. a ratio of about 6/5. Pop-up for children from 2 or 3 years old.

Ben Newman lives in the United Kingdom. He is an illustrator and artistic director. He participates in conferences at various universities in England and Europe. With his father, he loves painting and making 3D objects that they then display. He has produced work for a wide range of clients, including the Tate Modern, New York Times, BBC Radio 4, Google and The New Yorker.

13. Passchier, A., 2019. *Les contraires*. (« The opposites »). Kimane editions.
The book contains 10 pop-up images, 12 pages, format 18 cm x 20 cm, i.e. a ratio of 1/1. For children from 0 to 3 years old.

Anne Passchier is an illustrator and surface designer from the Netherlands, currently living in Pittsburgh, Pennsylvania, USA. She studied at the Ringling College of Art and Design in Florida. She illustrates children's books and designs decorative motifs for gift wrapping, notably for American Greetings.

14. Rouillac, P., 2015. *Créature fantastique*. (« Fantastic creature »). Paris: Éditions du Seuil Jeunesse.
The book consists of 7 large pop-up images, 18 pages, format 20 cm x 28 cm, i.e. a ratio of approximately 7/5.

Paul Rouillac, born in 1985, is a French artist.
A trained binder, he learned binding in London and discovered the world of paper. As a final project, he designed a mobile on the theme of wind, a 3 m^3 book inside which one can move.
Poet and book sculptor, he has designed several pop-ups.
In Bréhat, Brittany (France), he leads workshops and workshops for children and adults.

15. Rowling, J.K., Collective work, 2016. *Les Animaux fantastiques. Le Carnet magique de Nobert Dragonneau.* Paris: Gallimard jeunesse. Translated into French by Céline Grimault from the original *Fantastic Beasts and Where to Find Them*.
The book consists of 10 pop-up images, 48 pages, cardboard format 21 cm x 2 x 21 cm, i.e. a ratio of 1/1. This youth documentary is a pop-up for children from 8 to 18 years old.

Joanne Rowling is an English novelist and screenwriter born on July 31, 1965, in South Gloucestershire. She owes her worldwide fame to the Harry Potter series, whose novels, translated into nearly eighty languages, have sold more than 500 million copies worldwide.

16. Sabuda, R., 2009. *Peter Pan*. The book is based on the original work by James Matthew Barrie, *Peter Pan*. London. Paris: Seuil Jeunesse. It consists of 6 large pop-up images and 22 reduced pop-up images, 12 pages, format 21.5 cm x 26.5 cm, i.e. a ratio of approximately 6/5.

Robert Sabuda was born in 1965 in Michigan, United States. A graduate of the Pratt Institute in New York City, he is a painter in New York City and an illustrator of children's books, particularly animated books.

17. Sabuda, R., 2003. *Alice's Adventures in Wonderland*. The book is based on the original work by Lewis Carroll, *Alice's Adventures in Wonderland*. New York: Simon & Schuster Children's Publishing Division. Reprint 2003. New York: Little Simon Publishing.
The book is composed of 6 large pop-up images and 20 reduced pop-up images, 12 pages, format 21.5 cm x 26.5 cm, i.e. a ratio of approximately 6/5.

18. Lo Monaco, Duisit and others, collective work, 2018. *Le Petit Prince, Le Grand Livre Pop-Up d'Antoine de Saint-Exupéry*. First edition 2015, according to the text of the Gallimard editions of 1946. Paris: Gallimard Jeunesse.
The book consists of 24 pop-up images, 64 pages, format 18.7 x 26.6 cm, i.e. a ratio of approximately 7/5. Pop-up for children from 6 to 18 years old.

> Antoine de Saint-Exupéry, a French writer, poet, aviator and reporter, was born on 29 June 1900 in Lyon and disappeared in flight on 31 July 1944 off Marseille. In 1919, he enrolled as a free auditor in architecture at the École Nationale Supérieure des Beaux-Arts. From 1932 Saint-Exupéry devoted himself to journalism. The Little Prince, written in New York during the war, and illustrated with his own watercolours, was published in 1943 in New York, then posthumously in France by Gallimard in 1946.

19. Selena, E., 2018. *Jungle.* Paris: Albums Gallimard Jeunesse.
The book contains 6 pop-up images, 12 pages, format 22 cm x 27 cm, i.e. a ratio approximately 6/5. Pop-up for children from 4 to 9 years old.

> Elena Selena was born in Vilnius (Lithuania) in 1993. After first training at the Vilnius Academy of Fine Arts, she entered the Estienne School (ESAIG) in Paris and discovered the animated book. She creates many paper universes.

20. Ug, P., 2018. *Robopop.* Paris: Les grandes personnes éditions.
The book contains 5 pop-up images, 10 pages, format 10 cm x 20 cm, i.e. a ratio of 2/1.

> Philippe Ug, born in France in 1958, is a graphic designer and a graduate of the Duperré School of Applied Arts in Paris. He is a paper engineer, screen printer, printer and teacher. He produces pop-up books, in small series, and offers them at affordable prices in bookshops.

21. Ug, P., 2014. *Le jardin des papillons*. (« The Butterfly Garden »). Paris: Les grandes personnes éditions.
The book contains 7 large pop-up images, 18 pages, format 15 cm x 21 cm, i.e. a ratio of approximately 7/5.

Statistical summary drawn up on the basis of the cited reference works (Table 2)

Table 2
Pop up ratios: summarised statistical information on the works cited

	Illustrators	Nb Illust. (1)	Nb Pages (2)	Ratio illust./p (3)	Ratio L/w (4)	length L (5)	width w (6)	ages f rom, to (7)
1	Blackwell	10	28	36%	1.23	28	22.7	6 to 8
2	Bourgon	10	8	125%	1.39	25	18	0 to 3
3	Colombier	21	10	210%	1.00	18.1	18.1	1 to 3
4	Duisit	12	28	43 %	1,00	21	21	-
5	Duprat	21	56	38%	1.04	27.6	26.6	9 to 12
6	Ehrhard	10	24	42%	1.11	20.5	18.5	0 to 3
7	Fiorin	15	10	150%	1.26	28.4	22.5	0 to 5
8	Hawcock	6	14	43%	1.11	24.3	21.8	6 to 8
9	Hess	9	16	56%	1.18	31	26.3	-
10	Joffre	14	10	140%	1.00	17.5	17.5	1 to 3
11	Lo Monaco	10	17	59%	1.33	20	15	-
12	Newman	6	10	60%	1.17	28	24	2 to 3
13	Passchier	10	12	83%	1.11	20	18	0 to 3
14	Rouillac	7	18	39%	1,40	28	20	-
15	Rowling	10	48	21%	1.00	21	21	8 to 18
16	Sabuda 2009	28	12	233%	1.23	26.5	21.5	-
17	Sabuda 2003	26	12	217%	1.23	26.5	21.5	-
18	Saint-Exupéry	24	64	38%	1.42	26.6	18.7	6 to 18
19	Selena	6	12	50%	1.23	27	22	4 to 9
20	Ug 2018	5	10	50%	2.00	20	10	-
21	Ug 2014	7	18	39%	1.40	21	15	-

Note. Col. 1: number of illustrations; col 2: number of pages, col.3: ratio Illustrations/pages; Col. 4: format ratio length/width (L/w); col. 5: length (L); col. 6: width (w); col. 7: ages from … to …; '-' means not significant' (ns) or not available (n/a).

Table 2a

Ratio images/pages. Average 62% (ages 0 to 3 years)

	Illustrators	Nb Illust. (1)	Nb Pages (2)	Ratio illust./p (3)	Ratio L/w (4)	length L (5)	width w (6)	ages from, to (7)
6	Ehrhard	10	24	42%	1.1	20.5	18.5	0 - 3
12	Newman	6	10	60%	1.2	28.0	24.0	2 - 3
13	Passchier	10	12	83%	1.1	20.0	18.0	0 - 3
	Average	9	15	62%	1.1	22.8	20.2	1 - 3

Table 2b

Ratio images/pages. Average: 179% (age from 0 to 5 years and more)

	Illustrators	Nb Illust. (1)	Nb Pages (2)	Ratio illust./p (3)	Ratio L/w (4)	length L (5)	width w (6)	ages from, to (7)
2	Bourgon	10	8	125%	1.3	25.0	18.0	1 - 3
3	Colombier	21	10	210%	1.0	18.1	18.1	1 - 3
7	Fiorin	15	10	150%	1.3	28.4	22.5	0 - 5
10	Joffre	14	10	140%	1.0	17.5	17.5	1 - 3
16	Sabuda 2009	28	12	233%	1.2	26.5	21.5	-
17	Sabuda 2003	26	12	217%	1.2	26.5	21.5	-
	Average	19	10	179%	1.2	23.7	19.9	1 - 4

Table 2c
Ratio images/pages. Average: 48% (age not specified)

	Illustrators	Nb Illust. (1)	Nb Pages (2)	Ratio illust./p (3)	Ratio L/w (4)	length L (5)	width w (6)	ages from, to (7)
4	Duisit	12	28	43 %	1,0	21,0	21,0	-
9	Hess	9	16	56%	1.2	31.0	26.3	-
11	Lo Monaco	10	17	59%	1.3	20.0	15.0	-
14	Rouillac	7	18	39%	1.4	28.0	20.0	-
20	Ug 2018	5	10	50%	2.0	20.0	10.0	-
21	Ug 2014	7	18	39%	1.4	21.0	15.0	-
	Average	8	18	48%	1.4	23.5	17.9	-

Note. ‘-‘ means not significant’ (ns) or not available (n/a).

Table 2d
Ratio images/pages. Average: 42% (ages 4 to 12 years)

	Illustrators	Nb Illust. (1)	Nb Pages (2)	Ratio illust./p (3)	Ratio L/w (4)	length L (5)	width w (6)	ages from, to (7)
1	Blackwell	10	28	36%	1.2	28.0	22.7	6 to 8
5	Duprat	21	56	38%	1.0	27.6	26.6	9 to 12
8	Hawcock	6	14	43%	1.1	24.3	21.8	6 to 8
19	Selena	6	12	50%	1.2	27.0	22.0	4 to 9
	Average	11	28	42%	1.2	26.7	23.3	6 to 9

Table 2e

Ratio images/pages. Average: 29% (ages 6 to 18 years)

	Illustrators	Nb Illust.	Nb Pages	Ratio illust./p	Ratio L/w	length L	width w	ages from, to
15	Rowling	10	48	21%	1.0	21.0	21.0	8 - 18
18	St. Exupéry	24	64	38%	1.4	26.6	18.7	6 - 18
	Average	17	56	29%	1.2	23.8	19.9	7 - 18

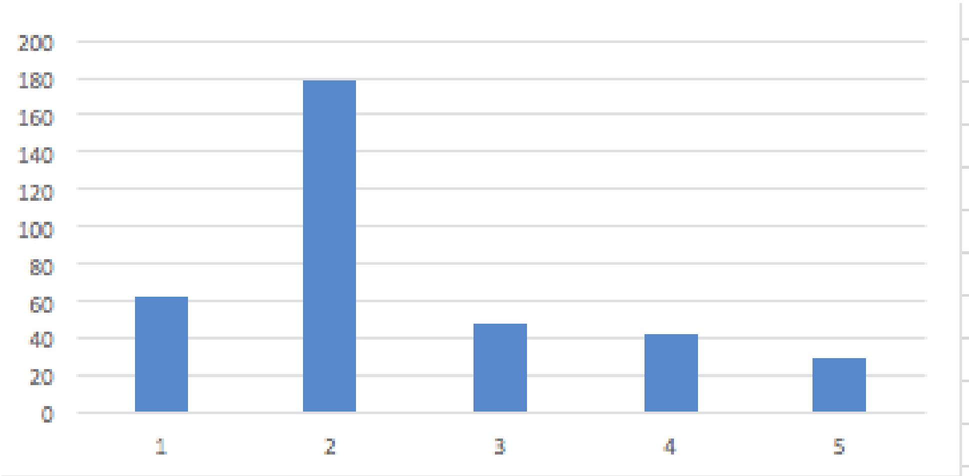
Table 2 f

Summary: Images/pages ratio (%)

Categories	Ratio Images/pages
1: tab. 2a	62%
2: tab. 2b	179%
3: tab. 2c	48%
4: tab. 2d	42%
5: tab. 2 e	29%

Graph 1

Images/pages ratio (%)



Categories:

1: from 0 to 3 years old; 2: around 4 years old and over; 3: all age groups; 4: about 5 to 9 years old; 5: about 6 to 18 years old.

C) Books consulted for the creation of the pop-ups

Carter, D. A. and J. Diaz, 1999. *The Element of Pop-up*. Texas, USA: White Heat Ltd, McKinney. Translated into French and adapted by M. Vesin and C. Baladi, 2015. *Pop-up, art et technique: créez vous-même des pop-up.* Toulouse: Éditions Milan et demi.

Keith A. F., 2013. *Technique de création de pop-up, les dessous de l'ingénierie papier*. Paris: Eyrolles.

Kyle, H. and U. Warchol, 2018. *The Art of the Fold*. London: Laurence King publishing. Translated by C. Reach, 2018. *L'art du pliage*. Paris: Pyramyd éditions.

D) Bibliography of 20 games under study (format of the games, weight, number of players, age and playing time)

1. *Giant Playing Cards*, 54 cards, 2020. Imported and designed by JJA, Le Blanc Mesnil, France. H.19 cm x L. 13 cm.

2. *50 Card Games*, 2018. Pack of cards, published by Igloo Books, Ltd., Cottage Farm, United Kingdom. Cribbage board, note pad and pencil, operating manuals, 64 pages, format 16 cm x 18.4 cm. Game box, 20 cm x 27.5 cm x 5.8 cm. For the whole family ('develop a winning strategy').

3. *Mastermind Junior*, 1994. Parker, Invicta Toys and Games Ltd. Counters, animals, instruction sheet: leaflet 6 pages, 14.9 cm x 26 cm. Game box: 15.9 cm x 30.9 cm x 5 cm, 2 players, 6 years old and over.

4. *Puzzle*. *Carte de France* (« Map of France »), 1991. Nathan games. 361 puzzles, 1 poster, 1 illustrated guide, 15 pages, format 13.4 cm x 21 cm, puzzle format: 36 cm x 49.5 cm, game box: 24 cm x 34.4 cm x 4 cm, age, 8 years and over.

5. *Scrabble*, 1948. J.W.S Spear &Sons, Ltd., UK, 1st edition 1948, and 1995 : 18-point rule of the game, unfolded apron 35.5 cm x 35.5 cm, game box: 18.5 x 36.5 cm, set for 2, 3 or 4 people, from 10 years and over.

6. *Scrabble, Spear's Games magnetic*, 1995. J.W.S Spear &Sons, Ltd., UK, 1995, 1 rule of the game, 4 pages, 8.7 cm x 15 cm, apron unfolded 16.9 cm x 19.4 cm, 102 letters, 4 magnetic bridges. Game box: 10.8 cm x 18 cm, 2 to 4 players, at 10 years old.

7. *Speed Game. 'Gare à la taupe'*, 2016. Janod Juratoys, France. Games rules: 2 pages, 10 cm x 12 cm, 1 board, 22.1 cm x 23, 7 cm, 4 stands, 20 pawns, 1 dice, 20 minutes ('speed, attention, concentration and strategy'), painted wooden pieces. Game box: 25 cm x 25 cm x 4.9 cm. For kids aged 3–6 years.

8. *Electronic, Minilab, Build your alarm*, 2017. Distributed by Buki France. Illustrated manual, 15 pages, 14.9 cm x 15 cm, platforms: 10 cm x 14 cm. Game box: 20 cm x 24.8 cm x 5 cm, 8 years old and over.

9. *Magic Beep Beep Beep. Le corps human* 2013. ('The Human Body'). Lisciani, Italy. Instruction book, 15 pages, 15.9 cm x 24 cm, 32 Quiz cards, 34 memo pieces, with the sound stethoscope, battery, tray 15.9 cm x 24 cm, weight 762 grams, game box 27.4 cm x 34 cm x 8 cm, 3 to 6 years old.

10. *Doctor Maboul,* 2015. Hasbro Gaming, UK, distributed in France. 1 manual page 21 cm x 29.5 cm batteries: 2 x AA – LR6, game board: 22.2 cm x 38.7 cm, 12 anatomical pieces, 12 cards, 1 player and more, game box: 25 cm x 40 cm x 4 cm, at 6 years old, game time: 15 minutes.

11. *La bonne paye*, 2002. Parker. Game rule, 4 pages 14.9 cm x 21 cm, 1 folding cardboard game board: 4 cardboard 25 cm x 25 cm x 0.9 cm (50.4 x 50, 4 cm unfolded), 6-counter, 23 acquisition cards, 16 loan cards, 50 mail cards, 23 event cards, 6 savings books, a set of tickets, 1 dice, game box: 26.9 cm x 40 cm x 5 cm, between 2 and 6 players, at 8 years old.

12. *Trivial Pursuit*, 1992. Junior Edition, Horn Abbot International Ltd, French Version, Parker. 2 pages game rules, 20 cm x 20 cm, 1 cardboard game board: 25.3 cm x 25.5 cm x 0.9 cm (unfolded: 50, 8 cm x 50, 8 cm), 1 dice, 1000 question-and-answer cards, 6 'camemberts' (or pawns), 36 marker triangles, game box: 26.8 cm x 26.8 cm x 8 cm, 2 to 6 players, 16–18 years old, game time: about 45 minutes (updated versions: 2400 questions), e.g., 'educational value: learning, sharing, having fun'.

13. *Ken Follet*. *The Pillars of the Earth*, 1989. London: McMillan. Translated into French by Jean Rosenthal, 1990: *Les piliers de la terre*, a game by Michael Rieneck & Stefan Stadler, illustrations: Michael Menzel. Game rule, 8 illustrated pages: 28 cm x 28 cm, cardboard game board 21.4 cm x 28.4 cm x 0.8 cm (unfolded: 43 cm x 56.9 cm), wooden cards, pawns and figurines, game box: 29.5 cm x 29.5 cm x 7 cm, 2 to 4 players, 12 years and over, game time 90 minutes, from a historical fiction by Ken Follett, written in 1989, and published by Macmillan, London.

14. *Checkers game*, 2017. Square Game, 40 wooden counters. Chequerboard of 40 cm x 40 cm, 998 grams. Number of players 2, at 3 years old.

15. *Chess game* (size N° 5, competition). Distribution VA variant, Paris, apron 45 cm x 45 cm x 1.3 cm, boxes 5 cm, pieces: king height 9 cm (base 3.5 cm). Pawn height 4.5 cm (base 2.5 cm), 32 chess pieces with storage box, player 2.

16. *Go game*. Aobo diffusion, The Art of enjoying, a 19 x 19 chipboard goban, convex ceramic stones, two bowls, apron 42.5 cm x 45.5 cm x 1.5 cm, tray weight: 1.6 kg, stone thickness: 0.6 cm, stone diameter: 2.2 cm, stone weight: 1.5 kg.

17. *Classic game set*, 2017. JeuJura, Saint Germain en Montagne, France. Booklet of 50 rules: Jeu des petits chevaux, Jeu de l'oie, Jeu de dames, Jeu de la marelle (« Little horses game, goose game, game of draughts, hopscotch game »), YAMS, 421. Wooden box: 33 cm x 33 cm x 5 cm, 880 grams, 2 to 6 players, 2 years and over.

18. *Hasbro Gaming Monopoly Game*, 2012. English version. Game box: 26.9 cm x 40.1 cm x 5.1 cm, 721 g, 2 to 6 players. For 8-12-year-olds.

19. *Cluedo. The Mystery Classic Game*, 2016. English version, distributed by Latest Bargains. Game box: 26.7 cm x 40 cm x 5 cm, 862 g, number of players 6, ages 8–18.

20. *Robomaker*, 2017. Clementoni Robomaker, Robotics Lab – Robot Construction Set, Fontenoce MC: Italy. Game box: 38 cm x 52.5 x 8 cm, 250 parts, 3 electric motors, 2 infrared sensors, 1 sensor, 1 loudspeaker, more than 250 components, 5 robots to programme manually or via an application, 2.2 kg, for 8–12 years.

Statistical summary drawn up on the basis of the cited games (Table 3)

Table 3
Summarised statistical information on the 20 games cited.

	Games	Number of pages	Format (w) book	Format (L) book	Ages years		Number of players	Time Minutes
1	Giant Playin Cards	ns	13.0	19.0	3	and +	2 - 6	ns
2	50 Card Games	64	16.0	18.4	3	and +	2 - 6	ns
3	Mastermind Junior (1994)	6	14.9	26.0	6	and +	2	ns
4	Puzzle Jeux Nathan (1991)	15	13,4	21,0	8	and +	1	ns
5	Scrabble (1948)	ns	ns	ns	10	and +	2 - 4	ns
6	Scrabble (1995)	4	8.7	15.0	10	and +	2 - 4	ns
7	Speed Game (2016)	2	10.0	12.0	3	to 6	3 - 6	20
8	Electronic (2018)	15	15.0	15.0	8	to 18	1	ns
9	Magique Bip Bip (2013)	15	15,9	24,0	3	to 6	1	ns
10	Docteur Maboul (2015)	1	21,0	29,5	6	and +	1	15
11	La bonne paye (2002)	4	14,9	21,0	8	and +	2 - 6	ns
12	Trivial Pursuit (1992)	2	20.0	20.0	16	to 18	2 - 6	45
13	Ken Follet (2007)	8	28,0	28,0	12	and +	2 - 4	90
14	Checkers game (2017)	ns	ns	ns	3	and +	2	ns
15	Chess game	ns	ns	ns	3	and +	2	ns
16	Go game	ns	ns	ns	4	and +	2	ns
17	Classic game (2017)	ns	ns	ns	2	and +	2 - 6	ns
18	Monopoly game (2012)	ns	ns	ns	8	to 12	2 - 6	ns
19	Cluedo (2016)	ns	ns	ns	8	to 18	6	ns
20	Robomaker (2017)	ns	ns	ns	8	to 12	1	ns

Note. Col. 1: Instruction book: number of pages; col.2: format of the book, width (w) and col. 3, length (L); col.4: ages from … to …; col. 5: number of players; col. 6: game time in minutes; 'ns' means 'not significant' or not available (n/a).

Table 3 continued.
Technical data

	Technical data	Weight (g)	Apron (w)	Apron (L)	Ap (h)	Box (w)	Box (L)	Box (h)
1	Giant Playin Cards	ns	ns	ns	ns	ns	ns	ns
2	50 Card Games	ns	ns	ns	ns	20.0	27.5	5.8
3	Mastermind Junior (1994)	ns	ns	ns	ns	15.9	30.9	5.0
4	Puzzle Jeux Nathan (1991)	ns	36,0	49,5	ns	24.0	34.4	4.0
5	Scrabble (1948)	ns	35.5	35.5	ns	18.4	36.4	ns
6	Scrabble (1995)	ns	16.9	19.4	ns	10.8	18.0	ns
7	Speed Game (2016)	ns	22.1	23.7	ns	25.0	25.0	4.9
8	Electronic (2018)	204	10.0	14.0	ns	20.0	25.0	5.0
9	Magique Bip Bip (2013)	762	15,9	24,0	ns	27.4	34.0	8.0
10	Docteur Maboul (2015)	ns	22,2	38,7	ns	25.0	40.0	4.0
11	La bonne paye (2002)	ns	50,5	50,4	ns	26.9	40.0	5.0
12	Trivial Pursuit (1992)	ns	50.8	50.8	0.9	26.8	26.8	8.0
13	Ken Follet (2007)	ns	43,0	56,9	na	29,5	29,5	7,0
14	Checkers game (2017)	998	40.0	40.0	ns	ns	ns	ns
15	Chess game	ns	45.0	45.0	1.3	ns	ns	ns
16	Go game	1600	42.5	45.5	1.5	ns	ns	ns
17	Classic game (2017)	880	ns	ns	ns	33.0	33.0	5.0
18	Monopoly game (2012)	721	ns	ns	ns	26.9	40.1	5.1
19	Cluedo (2016)	862	ns	ns	ns	26.7	40.0	5.0
20	Robomaker (2017)	2200	ns	ns	ns	38.0	52.5	8.0

Note. Col. 1: weight (g); col. 2: apron, width (w), col. 3:apron, length (L), col. 4: apron, height (h); col. 5: box, width (w), col. 6: box, length (L), col. 7: box, height (h). ‘ns’ means not significant or not available (n/a).

Table 3a
Booklets. Ratio Length/width (L/w)

Game rule booklets	Number pages	Format (w)	Format (L)	Ratio (L/w)
Giant Playing Cards	54	13.0	19.0	1.46
50 Card Games	64	16.0	18.4	1.15
Mastermind Junior (1994)	6	14.9	26.0	1.74
Puzzle Jeux Nathan (1991)	15	13,4	21,0	1,57
Speed Game (2016)	2	10.0	12.0	1.20
Electronic (2018)	15	15.0	15.0	1.00
Magique Bip Bip (2013)	15	15,9	24,0	1,51
Docteur Maboul (2015)	1	21,0	29,5	1,40
Trivial Pursuit (1992)	2	20.0	20.0	1.00
Ken Follet (2007)	8	28,0	28,0	1,00
La bonne paye (2002)	4	14,9	21,0	1,41
Average	17	16.6	21.3	1.31

Note. Booklets. Length (L), width (w). Ratio Length/width (L/w). Average: 1.31

Table 3b
Apron: Ratio Length /width (L/

Game apron format	Format (w)	Format (L)	Ratio (L/w)
Puzzle Jeux Nathan (1991)	36,0	49,5	1,38
Scrabble (1948)	35.5	35.5	1.00
Scrabble (1995)	16.9	19.4	1.15
Speed Game (2016),	22.5	24.0	1.07
Electronic (2018)	10.0	14.0	1.40
Magique Bip Bip (2013)	15,9	24,0	1,51
Docteur Maboul (2015)	22,2	38,7	1,74
La bonne paye (2002)	50,5	50,4	1,00
Trivial Pursuit (1992)	50.8	50.8	1.00
Ken Follet (2007)	43,0	56,9	1,32
Checkers game (2017)	40.0	40.0	1.00
Chess game	45.0	45.0	1.00

Go game	42.5	45.5	1.07
Monopoly	45.0	45.0	1.00
Average	34.0	38.5	1.19

Note. Apron:
Length (L), width (w).
Ratio Length /width (L/w).
Average: 1.19

Table 3c
Box: Ratio Length/width (L/w).

Box format	Format (w)	Format (L)	Ratio (L/w)
50 Card Games	20.0	27.5	1.38
Mastermind Junior (1994)	15.9	30.9	1.94
Puzzle Jeux Nathan (1991)	24,0	34,4	1,43
Scrabble (1948)	18.4	36.4	1.98
Scrabble (1995)	10.8	18.0	1.67
Speed Game (2016),	25.0	25.0	1.00
Electronic (2018)	20.0	25.0	1.25
Magique Bip Bip (2013),	27,4	34,0	1,24
Docteur Maboul (2015)	25,0	40,0	1,60
La bonne paye (2002),	26,9	40	1,49
Trivial Pursuit (1992)	26.8	26.8	1.00
Ken Follet (2007)	29,5	29,5	1,00
Classic game (2017)	33.0	33.0	1.00
Monopoly game (2012)	26.9	40.1	1.49
Cluedo (2016)	26.7	40.0	1.50
Robomaker (2017	38.0	52.5	1.38
Average	24.6	33.3	1.40

Note. format: Length (L), width (w).
Ratio Length /width (L/w).
Average: 1.4

Table 3d. Game box: Weight (g)

Weight (g)	(g)
Magique Bip Bip (2013),	762
Checkers game (2017)	998
Classic games (2017)	880
Go game	1600
Monopoly game (2012)	721
Cluedo (2016)	862
Robomaker (2017)	2200
Average	1146

Note. Weight (g). Average: 1146 g.

Table 3e. Game time

Game time (minutes)	minutes
Speed Game (2016)	20
Docteur Maboul (2015)	15
Trivial Pursuit (1992)	45
Ken Follet (2007)	90
Average	43

Note. Game time (minutes). Average: 43 minutes

E) Educational activity games for kids: logic games, brain games, mind challenge, games and word puzzles

Table 4

Educational activity games for kids

NB	Marketed by, year	Game name	Age, Material	Key skill and key learning outcomes
1	Skillmatics, 2017, 2018, 2019, Mumbai, India: Grasper Global Pvt. Ltd.	Mind Challenge	6 + years 23.6 x 19 x 2.4 cm 181 g 6 double-sided mat cards: 14.7 x 20.9 cm 1 pen and 1 duster cloth	Key skills: mental processing, strategising, problem solving, observation, concentration. Key learning outcomes: Thinking out of the box, solving equation, mental maths, vocabulary building, decoding, trial and error.

2	Skillmatics, 2017, 2018, 2019, Mumbai, India: Grasper Global Pvt. Ltd.	Brain Games	6 + years 23.6 x 19 x 2.4 cm 181 g 6 double-sided mat cards: 14.7 x 20.9 cm 1 pen and 1 duster cloth	Key skills: social and communication skills, strategising, problem solving, observation, concentration. Key learning outcomes: Thinking out of the box, solving equation, mental maths, decoding patterns, trial and error.
3	Skillmatics and Hamleys, on sale 2020, Mumbai, India: Grasper Global Pvt. Ltd. London: The Hamleys Group Ltd.	Logic Games	6 + years 23.6 x 19 x 2.4 cm 181 g 8 double-sided mat cards: 14.7 x 20.9 cm 1 pen and 1 duster cloth	Key skills: social and communication skills, strategising, decision-making, observation, concentration. Key learning outcomes: peer learning, vocabulary-building mental maths.
4	Youreka, Games & puzzles, Hamleys, on sale 2020, London: The Hamleys Group Ltd.	Make the word	3 + years Cardboard, Paper 20.5 x 14.5 x 3.5 cm 320 g 18 self-correcting 3-piece puzzles. 54 Puzzle Pieces	Key skills: problem-solving, concentration, imagination, memory, visual & mnemonic skills.
5	Youreka, Games & puzzles, Hamleys, on sale 2020, London: The Hamleys Group Ltd.	Youreka Opposites Puzzles	3 + years Cardboard, Paper 20.5 x 14.5 x 3.5 cm 320 g 24 self-correcting 2-piece puzzles. 48 Puzzle Pieces	Key skills: problem-solving, concentration, imagination, memory, visual & mnemonic skills. This game introduces children to the concept of opposites. The 24 self-correcting 2-piece puzzles ensure that the concept is learnt correctly from the start.

6	Ergo, 2020 Seattle, Washington: Catalyst Game Labs[157].	ERGO	12 + years 2–4 players Playing Time: 30 minutes 10.39 x 2.21 x 7.19 cm; 145.15 Grams	Rules of the game and rules of logic: "Do you exist? I think therefore I am? From Socrates to Descartes, the question has dogged mankind. Now with Ergo you can prove your existence while disproving the existence of your friends! I play therefore I am! " 4 of each Variable Card (A, B, C and D) 4 of each Operator Card (AND, OR, THEN) 6 NOT Cards 8 Parenthesis Cards 3 Fallacy Cards 3 Justification Cards 1 Tabula Rasa Card 1 Revolution Card 2 Wild Cards (1 Variable wild and 1 Operator wild) 3 Ergo Cards

[157] Game 6: Catalyst Game labs, 2015. Ergo. DriveThruRPG.com. Video demonstration, YouTube, Available at: <https://www.drivethrurpg.com/product/146618/Ergo> [Accessed 9 November 2020].

F) Thesis word count and personal illustrations inventory

THESIS

WORD COUNT AND PERSONAL ILLUSTRATIONS INVENTORY

GAME 1, 2 and 3. The Puzzles

Game 1: 10 puzzles
Game 2: 12 puzzles
Game 3: 15 puzzles
1 box of playing cards
60 recto verso playing cards
3 Booklets
1 Fibonacci puzzle
2 Pythagoras' theorem puzzles

The Logical Spring Game

1 pop-up
1 Booklet

The Cheshire Cat Paradox

1 pop-up
1 Booklet

The Venn Game

1 pop-up game board
30 wooden pieces
1 box of playing cards
24 recto verso playing cards
1 Booklet

The Square of opposition Game

1 game board
1 box of playing cards
64 recto verso playing cards
8 characters figures with their 8 associated playing cards
4 counters
4 coloured dice
1 Booklet

The Lewis Carroll Game

1 pop-up game board
14 character figures
1 box of playing cards
14 recto verso playing cards
30 counters
1 Booklet

The Robot Game

2 game boards with 1 pop-up game board
2 boxes of playing cards
1 box of 48 playing cards
1 box of 100 playing cards
200 counters
8 illustrated dice
4 Booklets

Total

12 games
40 puzzles
5 pop-ups
7 game boards
318 playing cards
234 counters
52 character figures/wooden pieces
12 dice
6 boxes of playing cards
12 Booklets

Word count

Turnitin and Words: 55,011 words in total

Abstract: 280 words
Title page, acknowledgements and text (without textboxes, footnotes and endnotes): 36,633 words
Annexes: 3450 words
Title page, acknowledgements, text (without textboxes, footnotes and endnotes) and annexes: 40083 words
Bibliography I: 5234 words
Bibliography II: 7058 words
Bibliographies: 12292 words